

BRANCHING RULES FOR UNRAMIFIED PRINCIPAL SERIES REPRESENTATIONS OF $GL(3)$ OVER A p -ADIC FIELD

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ABSTRACT. On restriction to the maximal compact subgroup $GL(3, \mathcal{R})$, an unramified principal series representation of the p -adic group $GL(3, F)$ decomposes into a direct sum of finite-dimensional irreducibles each appearing with finite multiplicity. We describe a coarser decomposition into components which, although reducible in general, capture the equivalences between the irreducible constituents.

1. INTRODUCTION

The aim of this paper is to investigate the relationship between the representation theory of a p -adic group G and its maximal compact subgroups K . Given an admissible representation of G , its restriction to K decomposes as a direct sum of smooth irreducible representations of K each with finite multiplicity. The problem of describing this decomposition when $G = GL(2, F)$ and $K = GL(2, \mathcal{R})$, for F a non-archimedean local field of odd residual characteristic and its ring of integers \mathcal{R} , was extensively studied by Silberger [8] and Casselman [3] with the restriction on the characteristic removed. Further, the case of the principal series representations for $G = SL(2, F)$ was considered by the second author in [6].

We are interested in $G = GL(3, F)$ and its unramified principal series representations; the ramified case will be treated in a separate paper. The restriction to $K = GL(3, \mathcal{R})$ of any unramified principal series representation is simply the permutation representation over the subgroup B of upper triangular matrices in K . In particular, this contains the pull-back of the corresponding permutation representation for the group $GL(3, \mathfrak{f})$, defined over the residue field \mathfrak{f} of F , and the decomposition of this is well known [9]: each irreducible constituent can be expressed as an alternating sum of permutation representations over certain standard parabolic subgroups in $GL(3, \mathfrak{f})$.

Our approach is generalise this by considering representations $V_{\mathbf{c}}$, indexed by triples $\mathbf{c} = (c_1, c_2, c_3)$ with $0 \leq c_1, c_2 \leq c_3 \leq c_1 + c_2$, which are expressible in

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terms of permutation representations over compact open subgroups containing B . By determining the double coset structure of K we are able to calculate the intertwining number $\mathcal{I}(V_c, V_d)$ for any two such components; that is, the dimension of the space of K -homomorphisms between V_c and V_d . Although V_c is irreducible when $c_3 = c_1 + c_2$ or $\max\{c_1, c_2\}$, we find that it is reducible in general with $\mathcal{I}(V_c, V_c)$ depending on the order of the residue field. However, it transpires that two components are either completely equivalent or contain no common constituents. In the final section, we present an application of our results to a family of virtual representations defined by Lees [5] as analogues of the Steinberg representation.

2. PRINCIPAL SERIES REPRESENTATIONS

Let F be a non-archimedean local field with ring of integers \mathcal{R} and residue field \mathfrak{f} . We will assume that \mathfrak{f} has odd characteristic and order q . If π denotes a conductor of F then the maximal ideal of \mathcal{R} is $\mathcal{P} = \pi\mathcal{R}$. For each positive integer $n \in \mathbb{Z}_+$ we define $\mathcal{P}^n = \{x \in F : \text{val}(x) \geq n\}$ where val is the discrete valuation on F normalised so that $\text{val}(\pi) = 1$.

Let $\mathbb{G} = \text{GL}(3)$, then $\mathbb{G}(F) = \text{GL}(3, F)$ is a locally compact group with maximal compact open subgroup $K = \mathbb{G}(\mathcal{R}) = \text{GL}(3, \mathcal{R})$. Indeed, the topology on $\mathbb{G}(F)$ has a neighbourhood base about the identity given by the compact open subgroups $K_n = 1 + M_{3,3}(\mathcal{P}^n)$ for $n \in \mathbb{Z}_+$. Further, let \mathbb{B} be the subgroup of upper triangular matrices and recall that \mathbb{B} decomposes as $\mathbb{B} = \mathbb{T}\mathbb{U}$ where \mathbb{T} is the subgroup of diagonal matrices and \mathbb{U} is the subgroup of upper unitriangular matrices. We will denote by B , T and U the subgroups $\mathbb{B}(\mathcal{R})$, $\mathbb{T}(\mathcal{R})$ and $\mathbb{U}(\mathcal{R})$ of K respectively.

Given a character χ of $\mathbb{T}(F)$ we may extend it to a character of $\mathbb{B}(F)$, again denoted χ , by defining it to be trivial on $\mathbb{U}(F)$. The corresponding principal series representation of $\mathbb{G}(F)$ is the induced representation $\text{Ind}_{\mathbb{B}(F)}^{\mathbb{G}(F)} \chi$ consisting of the space smooth functions

$$V = \{f \in C^\infty(\mathbb{G}(F)) : f(bg) = \chi(b)|b|f(g) \text{ for all } g \in \mathbb{G}(F), b \in \mathbb{B}(F)\}$$

with the action of $\mathbb{G}(F)$ given by right translation. The normalization factor $|b|$ is introduced to ensure that $\text{Ind}_{\mathbb{B}(F)}^{\mathbb{G}(F)} \chi \simeq \text{Ind}_{\mathbb{B}(F)}^{\mathbb{G}(F)} \chi'$ whenever χ and χ' lie in the same orbit under the Weyl group W of \mathbb{G} (see [2]*Theorem 3.3, for example).

We will be interested in the restriction of the principal series representation V to the maximal compact subgroup K . As $\mathbb{G}(F) = K\mathbb{B}(F)$ and $B = \mathbb{B}(F) \cap K$, Mackey theory implies that

$$\text{Res}_K^{\mathbb{G}(F)} V \simeq \text{Ind}_B^K \text{Res}_B^{\mathbb{B}(F)} \chi.$$

This can be interpreted as the principal series representation of K obtained from the character $\text{Res}_T^{\mathbb{T}(F)} \chi$ of T . The first step towards decomposing the restriction into irreducibles is the following result regarding the principal congruence subgroups K_n of K .

Lemma 2.1. *The subspaces V^{K_n} of vectors fixed under the action of K_n are K -stable and finite-dimensional. They are non-zero if and only if $K_n \cap T \subseteq \ker(\chi)$, in which case χ extends trivially to a character of BK_n and*

$$V^{K_n} = \text{Ind}_{BK_n}^K \chi$$

where both sides are viewed as K -representations.

In this paper we will be concerned the unramified principal series; that is, the case where the restriction of χ to T is the trivial character 1. Here we obtain the permutation representation

$$\text{Res}_K^{\mathbb{G}(F)} V \simeq \text{Ind}_B^K 1$$

and for each $n \in \mathbb{Z}_+$

$$V^{K_n} = \text{Ind}_{BK_n}^K 1.$$

3. A DECOMPOSITION

The filtration of K by congruence subgroups allows us to decompose the representation V into a direct sum of finite-dimensional K -invariant subspaces

$$V \simeq \bigoplus_{n=0}^{\infty} V^{K_n} / V^{K_{n-1}}.$$

However, these quotients are far from being irreducible in general so we will consider a finer filtration of K obtained from certain compact open subgroups $C_{\mathbf{c}}$.

Define the partially ordered set

$$\mathbf{T} = \{\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3 : 0 \leq c_1, c_2 \leq c_3 \leq c_1 + c_2\}$$

with order given by $\mathbf{c} \succeq \mathbf{d}$ if and only if $c_i \geq d_i$ for each i . We associate to each triple $\mathbf{c} = (c_1, c_2, c_3) \in \mathbf{T}$ a compact open subgroup $C_{\mathbf{c}}$ of K by defining

$$C_{\mathbf{c}} = \left[\begin{array}{ccc} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_1} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_3} & \mathcal{P}^{c_2} & \mathcal{R} \end{array} \right] \cap K$$

and note that $C_{\mathbf{c}} \subseteq C_{\mathbf{d}}$ if and only if $\mathbf{c} \succeq \mathbf{d}$. Consequently, if for each $\mathbf{c} \in \mathbf{T}$ we set

$$U_{\mathbf{c}} = \text{Ind}_{C_{\mathbf{c}}}^K 1$$

then $U_{\mathbf{d}}$ arises as a subrepresentation of $U_{\mathbf{c}}$ precisely when $\mathbf{d} \preceq \mathbf{c}$. Thus we can consider the quotient

$$V_{\mathbf{c}} = U_{\mathbf{c}} / \sum_{\mathbf{d} \prec \mathbf{c}} U_{\mathbf{d}}.$$

In particular, since $BK_n = C_{(n,n,n)}$, we see that

$$V^{K_n} = \bigoplus_{\mathbf{c} \preceq (n,n,n)} V_{\mathbf{c}}.$$

Our aim is to determine the reducibility of and equivalences between the $V_{\mathbf{c}}$. To achieve this we first give a description of $V_{\mathbf{c}}$ as an alternating sum in

the Grothendieck group $\mathcal{K}_0(K)$ of K . Recall that $\mathcal{K}_0(K)$ is the abelian group generated by the isomorphism classes $[V]$ of finitely-generated representations V of K together with the relations $[V \oplus U] = [V] + [U]$ and $[V/U] = [V] - [U]$.

We begin with some notation. Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbf{T}$ and, if c_1 and c_2 are both non-zero, for each $1 \leq i \leq 3$ define

$$\mathbf{c}_{\{i\}} = (c_1 - \delta_{i,1}, c_2 - \delta_{i,2}, c_3 - \delta_{i,3})$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. If $c_1 = 0$ then $\mathbf{c} = (0, c_2, c_3)$ and we only consider $\mathbf{c}_{\{3\}} = (0, c_2 - 1, c_3 - 1)$. Similarly, if $c_2 = 0$ then we only have $\mathbf{c}_{\{3\}} = (c_1 - 1, 0, c_3 - 1)$. The set of all triples in \mathbf{T} lying immediately below \mathbf{c} is then $\{\mathbf{c}_{\{i\}} : i \in S_{\mathbf{c}}\}$ where $S_{\mathbf{c}} = \{i : \mathbf{c}_{\{i\}} \in \mathbf{T}\}$. In particular, this means that

$$V_{\mathbf{c}} = U_{\mathbf{c}} / \sum_{i \in S_{\mathbf{c}}} U_{\mathbf{c}_{\{i\}}}.$$

Further, let $\mathbf{c}_{\emptyset} = \mathbf{c}$ and for each non-empty $I \subseteq S_{\mathbf{c}}$ define

$$\mathbf{c}_I = \max\{\mathbf{d} \in \mathbf{T} : \mathbf{d} \preceq \mathbf{c}_{\{i\}} \text{ for all } i \in I\}.$$

For example, if $\mathbf{c} = (2, 3, 4)$ then $\mathbf{c}_{\{1,2\}} = (1, 2, 3)$ since $(1, 2, 4) \notin \mathbf{T}$.

Lemma 3.1. *For each $I, J \subseteq S_{\mathbf{c}}$ we have $U_{\mathbf{c}_I} \cap U_{\mathbf{c}_J} = U_{\mathbf{c}_{I \cup J}}$.*

Lemma 3.2. *For any $\mathbf{c} \in \mathbf{T}$ with $S_{\mathbf{c}} = \{1, 2, 3\}$*

$$(U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}}) \cap U_{\mathbf{c}_{\{3\}}} = U_{\mathbf{c}_{\{1,3\}}} + U_{\mathbf{c}_{\{2,3\}}}.$$

Proof. This follows from [1]*Lemma 13 since $C_{\mathbf{c}_{\{i\}}} C_{\mathbf{c}_{\{j\}}} = C_{\mathbf{c}_{\{3\}}} C_{\mathbf{c}_{\{i\}}} = C_{\mathbf{c}_{\{i,j\}}}$ for each i . \square

Proposition 3.3. *For any $\mathbf{c} \in \mathbf{T}$*

$$[V_{\mathbf{c}}] = \sum_{I \subseteq S_{\mathbf{c}}} (-1)^{|I|} [U_{\mathbf{c}_I}].$$

Proof. First note that if $S_{\mathbf{c}} = \{i\}$ then $V_{\mathbf{c}} = U_{\mathbf{c}}/U_{\mathbf{c}_i}$ so clearly

$$[V_{\mathbf{c}}] = [U_{\mathbf{c}}] - [U_{\mathbf{c}_{\{i\}}}]$$

Further, if $S_{\mathbf{c}} = \{i, j\}$ then $U_{\mathbf{c}_{\{i\}}} + U_{\mathbf{c}_{\{j\}}} = X \oplus U_{\mathbf{c}_{\{j\}}}$ where $X = U_{\mathbf{c}_{\{i\}}}/(U_{\mathbf{c}_{\{i\}}} \cap U_{\mathbf{c}_{\{j\}}})$ and $U_{\mathbf{c}_{\{i\}}} \cap U_{\mathbf{c}_{\{j\}}} = U_{\mathbf{c}_{\{i,j\}}}$. This gives $[U_{\mathbf{c}_{\{i\}}} + U_{\mathbf{c}_{\{j\}}}] = [U_{\mathbf{c}_{\{i\}}}] + [U_{\mathbf{c}_{\{j\}}}] - [U_{\mathbf{c}_{\{i,j\}}}]$ and $V_{\mathbf{c}} = U_{\mathbf{c}}/(U_{\mathbf{c}_{\{i\}}} + U_{\mathbf{c}_{\{j\}}})$ implies that

$$[V_{\mathbf{c}}] = [U_{\mathbf{c}}] - [U_{\mathbf{c}_{\{i\}}}] - [U_{\mathbf{c}_{\{j\}}}] + [U_{\mathbf{c}_{\{i,j\}}}]$$

Finally, if $S_{\mathbf{c}} = \{1, 2, 3\}$ then $U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}} + U_{\mathbf{c}_{\{3\}}} = X \oplus U_{\mathbf{c}_{\{3\}}}$ where on this occasion $X = (U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}})/((U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}}) \cap U_{\mathbf{c}_{\{3\}}})$. From Lemma 3.2 we know that $(U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}}) \cap U_{\mathbf{c}_{\{3\}}} = U_{\mathbf{c}_{\{1,3\}}} + U_{\mathbf{c}_{\{2,3\}}}$ so using the same argument as before we see that

$$[X] = [U_{\mathbf{c}_{\{1\}}}] + [U_{\mathbf{c}_{\{2\}}}] - [U_{\mathbf{c}_{\{1,2\}}}] - [U_{\mathbf{c}_{\{1,3\}}}] - [U_{\mathbf{c}_{\{2,3\}}}] + [U_{\mathbf{c}_{\{1,2,3\}}}]$$

Hence, $V_{\mathbf{c}} = U_{\mathbf{c}}/(U_{\mathbf{c}_{\{1\}}} + U_{\mathbf{c}_{\{2\}}} + U_{\mathbf{c}_{\{3\}}})$ gives

$$[V_{\mathbf{c}}] = [U_{\mathbf{c}}] - [U_{\mathbf{c}_{\{1\}}}] - [U_{\mathbf{c}_{\{2\}}}] - [U_{\mathbf{c}_{\{3\}}}] + [U_{\mathbf{c}_{\{1,2\}}}] + [U_{\mathbf{c}_{\{1,3\}}}] + [U_{\mathbf{c}_{\{2,3\}}}] - [U_{\mathbf{c}_{\{1,2,3\}}}]$$

as required. \square

The space of K_1 -fixed vectors in V

$$V^{K_1} = \text{Ind}_{C_{(1,1,1)}}^K 1$$

is the pull-back to K of the permutation representation $\text{Ind}_{\mathbb{B}(f)}^{\mathbb{G}(f)} 1$ so its decomposition into irreducibles is well known. Specifically,

$$V^{K_1} = V_{(0,0,0)} \oplus V_{(0,1,1)} \oplus V_{(1,0,1)} \oplus V_{(1,1,1)}$$

where $[V_{(0,0,0)}] = [U_{(0,0,0)}]$ is the trivial representation; $[V_{(0,1,1)}] = [U_{(0,1,1)}] - [U_{(0,0,0)}]$ and $[V_{(1,0,1)}] = [U_{(1,0,1)}] - [U_{(0,0,0)}]$ are the equivalent irreducible constituents; and $[V_{(1,1,1)}] = [U_{(1,1,1)}] - [U_{(0,1,1)}] - [U_{(1,0,1)}] + [U_{(0,0,0)}]$ corresponds to the Steinberg representation which is irreducible with multiplicity 1.

More generally, we can use Proposition 3.3 to calculate the intertwining number between two quotients V_c and V_d as an alternating sum involving the intertwining numbers between various U_c and U_d

$$\mathcal{I}(V_c, V_d) = \sum_{I \subseteq S_c, J \subseteq S_d} (-1)^{|I|+|J|} \mathcal{I}(U_{c_I}, U_{d_J}).$$

However, since U_{c_I} and U_{d_J} are the permutation representations on C_{c_I} and C_{d_J} respectively, we have $\mathcal{I}(U_{c_I}, U_{d_J}) = |C_{c_I} \backslash K / C_{d_J}|$, the number of (C_{c_I}, C_{d_J}) -double cosets in K . Thus we obtain the following.

Corollary 3.4. *Let $c, d \in T$, then*

$$\mathcal{I}(V_c, V_d) = \sum_{I \subseteq S_c, J \subseteq S_d} (-1)^{|I|+|J|} |C_{c_I} \backslash K / C_{d_J}|.$$

Finally, we note that Proposition 3.3 also allows us to determine the dimensions of the V_c for $c \in T$ with $c_3 > 1$. If we let $|c| = c_1 + c_2 + c_3$ then

$$\dim U_c = [K : C_c] = \begin{cases} (q+1)(q^2+q+1)q^{|c|-3} & \text{if } c_1, c_2 > 0; \\ (q^2+q+1)q^{|c|-2} & \text{if } c_1 = 0 \text{ or } c_2 = 0 \end{cases}$$

so, writing $c_3 = c_1 + c_2 - k$ for some $0 \leq k \leq \min\{c_1, c_2\}$, we have

$$\dim V_c = \begin{cases} (q-1)(q+1)(q^2+q+1)q^{|c|-4} & \text{if } k = 0; \\ (q-1)(q-2)(q+1)(q^2+q+1)q^{|c|-5} & \text{if } k = 1; \\ (q-1)^3(q+1)(q^2+q+1)q^{|c|-6} & \text{if } 1 < k < \min\{c_1, c_2\}; \\ (q-1)^2(q+1)(q^2+q+1)q^{|c|-5} & \text{if } k = \min\{c_1, c_2\}. \end{cases}$$

4. (B,B)-DOUBLE COSETS

It is clear from Corollary 3.4 that we need to describe the (C_c, C_d) -double coset structure of K . However, before tackling the general case we examine the double cosets of K with respect to the subgroup B of upper triangular matrices. These, and indeed the double cosets in the case where $c = (c, c, c) = d$, have recently been described by Onn, Prasad and Vaserstein [7].

Let $W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, w_0\}$ denote the group of permutation matrices in K where s_i corresponds to the transposition $(i \ i+1)$ and w_0 is the element

of maximal length. From the Bruhat decomposition of $\mathrm{GL}(3, f)$ we can choose our (B, B) -double coset representatives to be of the form wk for some $w \in W$ and $k \in K_1$. If we let U^- denote the subgroup of lower unitriangular matrices in K , then the decomposition $K_1 = (K_1 \cap U^-)(K \cap B)$ means that we may take $k \in K_1 \cap U^-$. Further, we have $U^- = V_w^- V_w$ where

$$V_w = \{[k_{ij}] \in U^- : k_{ij} = 0 \text{ if } w(i) < w(j)\}$$

and

$$V_w^- = \{[k_{ij}] \in U^- : k_{ij} = 0 \text{ if } w(i) > w(j)\}.$$

Thus, writing $k = k_1 k_2$ with $k_1 \in V_w^-$, $k_2 \in V_w$ we see that $BwkB = Bwk_2B$ since $wk_1w^{-1} \in U$. We have therefore obtained the following special case of [4]*Proposition 2.6.

Lemma 4.1. *Every (B, B) -double coset representative in K can be chosen of the form wk for some $w \in W$ and $k \in V_w$.*

While Lemma 4.1 shows that there is exactly one double coset corresponding to w_0 , it does not give any information about the double cosets lying in the Iwahori subgroup BK_1 . Let $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$ with the convention that $a < \infty$ and $\infty + a = \infty - a = \infty$ for every $a \in \mathbb{Z}_+$. Define the set of triples

$$\mathbf{T}^\infty = \{(a_1, a_2, a_3) \in \overline{\mathbb{Z}}_+^3 : a_1, a_2 \leq a_3\}$$

and for each $\mathbf{a} \in \mathbf{T}^\infty$, $x \in \mathbb{R}^\times$ consider the element

$$t_{\mathbf{a}, x} = \begin{bmatrix} 1 & 0 & 0 \\ \pi^{a_1} & 1 & 0 \\ \pi^{a_3} x & \pi^{a_2} & 1 \end{bmatrix}$$

where we take $\pi^\infty = 0$ and so set $\mathrm{val}(0) = \infty$. In the following we identify $\mathcal{R}/\mathcal{P}^i$ with a set of representatives in \mathcal{R} chosen so that they contain the representatives corresponding to $\mathcal{R}/\mathcal{P}^j$ for each $j < i$.

Proposition 4.2. *A complete set of (B, B) -double coset representatives in BK_1 is*

$$\mathbf{R}^1 = \{t_{\mathbf{a}, x} : \mathbf{a} \in \mathbf{T}^\infty, x \in \mathbf{X}^{\mathbf{a}}\}$$

where

$$\mathbf{X}^{\mathbf{a}} = \begin{cases} \{1\} & \text{if } a_3 = \infty; \\ (\mathcal{R}/\mathcal{P}^{\min\{a_1, a_2, a_3 - a_1, a_3 - a_2\}})^\times & \text{if } a_1 + a_2 \neq a_3 \text{ and } a_3 < \infty; \\ \bigcup_{i=0}^\infty (1 + \pi^i \mathcal{R}^\times) \cap (\mathcal{R}/\mathcal{P}^{\min\{a_1, a_2\} + i})^\times & \text{if } a_1 + a_2 = a_3 \text{ and } a_3 < \infty. \end{cases}$$

Proof. From Lemma 4.1 we can choose our representative $t = [t_{ij}]$ to lie in $K_1 \cap U^-$. Indeed, since left and right multiplication by elements of B allows us to add multiples of t_{31} to t_{21} and t_{32} , we may assume that the lower triangular entries of t are such that $\max\{\mathrm{val}(t_{21}), \mathrm{val}(t_{32})\} \leq \mathrm{val}(t_{31})$. Further, conjugating by elements of T enables us to independently scale t_{21} and t_{32} by elements of \mathbb{R}^\times . We therefore obtain a representative of the form $t_{\mathbf{a}, x}$ for some $\mathbf{a} \in \mathbf{T}^\infty$ and $x \in \mathbb{R}^\times$.

To show that different triples from T^∞ correspond to different double cosets suppose that $g = [g_{ij}]$ and $g' = [g'_{ij}]$ are elements of B with $gt_{\mathbf{a},x} = t_{\mathbf{b},y}g'$ for some $\mathbf{a}, \mathbf{b} \in T^\infty$ and $x, y \in \mathcal{R}^\times$. The lower triangular entries give the equations

$$\begin{aligned}\pi^{a_1}g_{22} + \pi^{a_3}xg_{23} &= \pi^{b_1}g'_{11} \\ \pi^{a_2}g_{33} &= \pi^{b_3}g'_{12}y + \pi^{b_2}g'_{22} \\ \pi^{a_3}xg_{33} &= \pi^{b_3}yg'_{11}.\end{aligned}$$

The third equation clearly implies that $a_3 = b_3$ while the first equation gives $a_1 \leq b_1$ with $a_1 = b_1$ whenever $a_1 \neq a_3$. However, if $a_1 = a_3$ then $b_3 \geq b_1 \geq a_1 = a_3 = b_3$ and again $b_1 = a_1$. Similarly, $a_2 = b_2$ from the second equation so $\mathbf{a} = \mathbf{b}$.

We now fix an $\mathbf{a} \in T^\infty$ and address the admissible range of values for x . If $a_3 = \infty$ then $\pi^{a_3} = 0$ and it is clear that we may take $\mathbf{X}^{\mathbf{a}} = \{1\}$ so we will assume that $a_3 < \infty$. Let $x, y \in \mathcal{R}^\times$ and suppose that we are able to choose elements $g_{11}, g_{22}, g_{33} \in \mathcal{R}^\times$ and $g_{12}, g_{13}, g_{23} \in \mathcal{R}$ in such a way that the following three equations hold:

$$\begin{aligned}(1) \quad g_{11} &= g_{22} - \pi^{a_1}g_{12} - \pi^{a_3}xg_{13} + \pi^{a_3-a_1}xg_{23} \\ (2) \quad g_{33} &= g_{22} - \pi^{a_1}g_{12} + \pi^{a_2}g_{23} - \pi^{a_1+a_2}g_{13} + \pi^{a_3}yg_{13} + \pi^{a_3-a_2}yg_{12} \\ (3) \quad (x-y)g_{22} &= (\pi^{-a_2}g_{12} - \pi^{-a_1}g_{23} + g_{13})x(\pi^{a_1+a_2} - \pi^{a_3}y).\end{aligned}$$

Then setting

$$\begin{aligned}g'_{11} &= g_{11} + \pi^{a_1}g_{12} + \pi^{a_3}xg_{13} & g'_{12} &= g_{12} + \pi^{a_2}g_{13} \\ g'_{22} &= g_{22} - \pi^{a_1}g_{12} + \pi^{a_2}g_{23} - \pi^{a_1+a_2}g_{13} & g'_{13} &= g_{13} \\ g'_{33} &= g_{33} - \pi^{a_2}g_{23} + \pi^{a_1+a_2}g_{13} - \pi^{a_3}yg_{13} & g'_{23} &= g_{23} - \pi^{a_1}g_{13}\end{aligned}$$

gives elements $g = [g_{ij}]$ and $g' = [g'_{ij}]$ of B with

$$gt_{\mathbf{a},x} = t_{\mathbf{a},y}g'.$$

On the other hand, given $x, y \in \mathcal{R}^\times$ we see that if $g = [g_{ij}]$ and $g' = [g'_{ij}]$ are elements of B with $gt_{\mathbf{a},x} = t_{\mathbf{a},y}g'$ then (1–3) hold. Hence $t_{\mathbf{a},x}$ and $t_{\mathbf{a},y}$ represent the same double coset precisely when such solutions exist.

First suppose that $a_1 + a_2 \neq a_3$. If $t_{\mathbf{a},x}$ and $t_{\mathbf{a},y}$ represent the same double coset for distinct $x, y \in \mathcal{R}^\times$ then from (3) we see that

$$(4) \quad \text{val}(x-y) \geq \min\{a_1, a_2, a_3 - a_1, a_3 - a_2\}.$$

Conversely, suppose that we have distinct $x, y \in \mathcal{R}^\times$ so that (4) holds. If the minimum occurs for a_1 then $a_3 - a_2 > a_1$, since $a_1 + a_2 \neq a_3$, and we have $\text{val}(\pi^{a_1+a_2} - \pi^{a_3}y) = a_1 + a_2$. Setting g_{23} and g_{13} both to be zero and choosing g_{12} with $\text{val}(g_{12}) = \text{val}(x-y) = a_1$ will give $g_{22} \in \mathcal{R}^\times$ by (3). Further, $g_{11}, g_{33} \in \mathcal{R}^\times$ by (1) and (2) since $a_3 - a_2 > 1$. If the minimum occurs for $a_3 - a_1$ then $a_3 - a_1 < a_2$ and $\text{val}(\pi^{a_1+a_2} - \pi^{a_3}y) = a_3$. Taking g_{12} and g_{13} to be zero and g_{23} such that $\text{val}(g_{23}) = \text{val}(x-y) - (a_3 - a_1)$ gives $g_{22} \in \mathcal{R}^\times$ and, provided that we make the specific choice $g_{23} = y - x$ when $a_3 = a_1$, we will also have $g_{11}, g_{33} \in \mathcal{R}^\times$. The arguments when the minimum is a_2 or $a_3 - a_2$ are similar and so we obtain a solution of (1–3) in each case. Hence for every

$\mathbf{a} \in \mathbb{T}^\infty$ with $a_1 + a_2 \neq a_3$ we may take a representative $t_{\mathbf{a},x}$ with x lying in the set

$$\mathbf{X}^{\mathbf{a}} = \mathcal{R}/\mathcal{P}^{\min\{a_1, a_2, a_3 - a_1, a_3 - a_2\}}$$

and distinct elements of this set give distinct double cosets.

Now suppose that $a_1 + a_2 = a_3$ and that $t_{\mathbf{a},x}$ and $t_{\mathbf{a},y}$ represent the same double coset for distinct elements $x, y \in \mathcal{R}^\times$. If $\text{val}(1-x) > \text{val}(1-y)$ then $\text{val}(x-y) = \text{val}((1-y) - (1-x)) = \text{val}(1-y)$ and if $\text{val}(1-x) < \text{val}(1-y)$ then $\text{val}(x-y) = \text{val}(1-x) < \text{val}(1-y)$. However, from (3) we know that $\text{val}(x-y) \geq \min\{a_1, a_2\} + \text{val}(1-y) > \text{val}(1-y)$. Therefore, we must have

$$(5) \quad \text{val}(1-x) = \text{val}(1-y) = i \quad \text{and} \quad \text{val}(x-y) \geq \min\{a_1, a_2\} + i.$$

Conversely, let $x, y \in \mathcal{R}^\times$ be such that condition (5) holds. If $a_1 \leq a_2$ then choosing $g_{23} = g_{13} = 0$ and g_{12} with $\text{val}(g_{12}) = \text{val}(x-y) - a_1 - i$ gives $g_{22} \in \mathcal{R}^\times$ and $g_{11}, g_{33} \in \mathcal{R}^\times$ since $a_3 - a_2 = a_1 > 0$. If $a_2 \leq a_1$ there is a similar argument and we again have a solution of (1-3) in each case. Hence for every $\mathbf{a} \in \mathbb{T}^\infty$ with $a_1 + a_2 = a_3$ we may take a representative $t_{\mathbf{a},x}$ with x from the set

$$\mathbf{X}^{\mathbf{a}} = \bigcup_{i=0}^{\infty} (1 + \pi^i \mathcal{R}^\times) \cap (\mathcal{R}/\mathcal{P}^{\min\{a_1, a_2\} + i})^\times$$

and distinct elements of this set give distinct double cosets. \square

Theorem 4.3. *A complete set of (B, B) -double cosets in K is given by*

$$\mathbf{R} = \{t_{\mathbf{a},x}, s_1^{(\alpha,\beta)}, s_2^{(\alpha,\beta)}, s_1 s_2^{(\alpha)}, s_2 s_1^{(\alpha)}, w_0 : \mathbf{a} \in \mathbb{T}^\infty, x \in \mathbf{X}^{\mathbf{a}}, \alpha, \beta \in \overline{\mathbb{Z}}_+\}$$

where

$$t_{\mathbf{a},x} = \begin{bmatrix} 1 & 0 & 0 \\ \pi^{a_1} & 1 & 0 \\ \pi^{a_3 x} & \pi^{a_2} & 1 \end{bmatrix}, \quad s_1^{(\alpha,\beta)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \pi^\beta & \pi^\alpha & 1 \end{bmatrix}, \quad s_2^{(\alpha,\beta)} = \begin{bmatrix} 1 & 0 & 0 \\ \pi^\beta & 0 & 1 \\ \pi^\alpha & 1 & 0 \end{bmatrix},$$

$$s_1 s_2^{(\alpha)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \pi^\alpha & 1 & 0 \end{bmatrix}, \quad s_2 s_1^{(\beta)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \pi^\alpha & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad w_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof. We have shown that each double coset has a representative of the form wk for some $w \in W$, $k \in V_w$ and that, in particular, when $w = 1$ we can take it to be $t_{\mathbf{a},x}$ with $\mathbf{a} \in \mathbb{T}^\infty$, $x \in \mathbf{X}^{\mathbf{a}}$. If $w \neq 1$ then k has at most two non-zero entries below the diagonal and we are able to independently scale these by any element of \mathcal{R}^\times via left and right multiplication by T . This means that each representative can be chosen from the set \mathbf{R} described above.

Representatives associated to distinct Weyl group elements must give distinct double cosets by the Bruhat decomposition of $\text{GL}(3, \mathbb{f})$. Further, by Proposition 4.2 we know that distinct elements from $\mathbf{R}^1 = \{t_{\mathbf{a},x} : \mathbf{a} \in \mathbb{T}^\infty, x \in \mathbf{X}^{\mathbf{a}}\}$ give distinct double cosets. Thus we need to show that different representatives from \mathbf{R} with the same non-trivial Weyl group element give different double cosets. We will prove only the case when $w = s_1$ and remark that the remaining cases are analogous.

If $s_1^{(\alpha, \beta)}$ and $s_1^{(\alpha', \beta')}$ represent the same double coset for some $\alpha, \alpha', \beta, \beta' \in \overline{\mathbb{Z}}_+$ then there must be elements $g = [g_{ij}]$ and $g' = [g'_{ij}]$ of B with $gs_1^{(\alpha, \beta)} = s_1^{(\alpha', \beta')}g'$. This implies that the following two equations hold:

$$(6) \quad \pi^{\alpha'} g_{11} = \pi^{\alpha} g_{33} - \pi^{\alpha+\alpha'} g_{13} - \pi^{\alpha+\beta'} g_{23}$$

$$(7) \quad \pi^{\beta'} g_{22} = \pi^{\beta} g_{33} - \pi^{\beta+\beta'} g_{23}.$$

However (6) implies that $\alpha = \alpha'$ and (7) implies that $\beta = \beta'$. Hence if the pairs (α, β) and (α', β') are distinct then $s_1^{(\alpha, \beta)}$ and $s_1^{(\alpha', \beta')}$ represent different double cosets. \square

5. GENERAL DOUBLE COSETS

We now turn our attention to the case of (C_c, C_d) -double cosets for $c, d \in T$. In this situation it is possible for the image of C_c or C_d in $GL(3, f)$ to be a proper parabolic subgroup and so different Weyl group elements could represent the same double coset. To eliminate these duplications we introduce the subset $W_{c,d}$ of W defined as follows:

- (i) $W_{c,d} = \{1, s_1, s_2, s_1 s_2, s_2 s_1, w_0\}$ if $c, d \succeq (1, 1, 1)$;
- (ii) $W_{c,d} = \{1, s_1, w_0\}$ if $c = (c, 0, c)$ with $c > 0$ and $d \succeq (1, 1, 1)$, or vice versa;
- (iii) $W_{c,d} = \{1, s_2, w_0\}$ if $c = (0, c, c)$ with $c > 0$ and $d \succeq (1, 1, 1)$, or vice versa;
- (iv) $W_{c,d} = \{1, w_0\}$ if $c = (c, 0, c)$ or $(0, c, c)$ and $d = (d, 0, d)$ or $(0, d, d)$;
- (v) $W_{c,d} = \{1\}$ if $c = (0, 0, 0)$ or $d = (0, 0, 0)$.

Since $W_{c,d}$ forms a set of representatives for the corresponding double cosets in $GL(3, f)$ this ensures that representatives associated to distinct elements of $W_{c,d}$ will indeed yield distinct double cosets. We therefore need to identify a set $R_{c,d}^w$ of representatives associated to each $w \in W_{c,d}$. As in the previous section, we begin by looking at the set $R_{c,d}^1$ of representatives corresponding to the trivial element of W .

Definition 5.1. Define the set of triples

$$T^1 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : 1 \leq a_1, a_2 \leq a_3\}$$

and for any $c, d \in T$ let

$$(8) \quad T_{c,d} = \{a \in T^1 : a \preceq \underline{c}, a \preceq \underline{d} \text{ and } a_3 \leq \min\{a_1 + \underline{c}_2, \underline{d}_1 + a_2\}\}$$

with the following exceptions:

$$(9) \quad T_{c,d} = \begin{cases} \{(1, 1, 1)\} & \text{if } c = (0, 0, 0) \text{ or } d = (0, 0, 0); \\ \{(1, a, a) : a \leq \min\{c_2, d_2\}\} & \text{if } c_1 = d_1 = 0 \text{ and } c_2, d_2 > 0; \\ \{(a, 1, a) : a \leq \min\{c_1, d_1\}\} & \text{if } c_2 = d_2 = 0 \text{ and } c_1, d_1 > 0. \end{cases}$$

Here $\underline{c} = (\underline{c}_1, \underline{c}_2, \underline{c}_3)$ where $\underline{c}_i = \max\{c_i, 1\}$ for each i .

Lemma 5.2. *Let $c, d \in T$, then each $t_{a,x} \in R_{c,d}^1$ may be chosen with $a \in T_{c,d}$. Moreover, if $a, b \in T_{c,d}$ are distinct then $t_{a,x}$ and $t_{b,y}$ represent distinct double cosets.*

Proof. It is clear from Theorem 4.3 that $\mathbf{R}_{\mathbf{c},\mathbf{d}}^1$ can be taken to be a subset of $\{t_{\mathbf{a},x} : \mathbf{a} \in \mathbf{T}^1, x \in \mathcal{R}^\times\}$. One can show explicitly that for any $\mathbf{a} \in \mathbf{T}^1$ and $x \in \mathcal{R}^\times$ the double coset $C_c t_{\mathbf{a},x} C_d$ contains $t_{\mathbf{b},y}$ where $\mathbf{b} \preceq \mathbf{a}$ is defined by

$$\begin{aligned} b_1 &= \min\{a_1, \underline{c}_1, \underline{d}_1\}, \\ b_2 &= \min\{a_2, \underline{c}_2, \underline{d}_2\}, \\ b_3 &= \min\{a_3, c_3, d_3, a_1 + \underline{c}_2, \underline{d}_1 + a_2\}. \end{aligned}$$

Thus, all double coset representatives in $\mathbf{R}_{\mathbf{c},\mathbf{d}}^1$ may be chosen with \mathbf{a} in the set defined by (8). When $c_1 = d_1 = 0$ we can replace $\underline{d}_1 + a_2$ by a_2 in the definition of b_3 and, similarly, when $c_2 = d_2 = 0$ we can replace $a_1 + \underline{c}_2$ by a_1 . In these exceptional cases we may therefore choose \mathbf{a} from one of the sets given in (9).

We wish to show that distinct triples \mathbf{a} and \mathbf{b} from $\mathbf{T}_{\mathbf{c},\mathbf{d}}$ yield distinct double cosets so suppose that $g = [g_{ij}] \in C_c$ and $g' = [g'_{ij}] \in C_d$ are such that $gt_{\mathbf{a},x} = t_{\mathbf{b},y}g'$ for some $x, y \in \mathcal{R}^\times$. Write $g_{21} = \gamma_{21}\pi^{c_1}$, $g'_{21} = \gamma'_{21}\pi^{d_1}$, $g_{32} = \gamma_{32}\pi^{c_2}$, $g'_{32} = \gamma'_{32}\pi^{d_2}$, $g_{31} = \gamma_{31}\pi^{c_3}$ and $g'_{31} = \gamma'_{31}\pi^{d_3}$ where $\gamma_{ij}, \gamma'_{ij} \in \mathcal{R}$. Comparing the lower triangular elements in the above product gives the following three equalities:

$$(10) \quad \gamma_{21}\pi^{c_1} + g_{22}\pi^{a_1} + g_{23}x\pi^{a_3} = g'_{11}\pi^{b_1} + \gamma'_{21}\pi^{d_1}$$

$$(11) \quad \gamma_{32}\pi^{c_2} + g_{33}\pi^{a_2} = g'_{12}y\pi^{b_3} + g'_{22}\pi^{b_2} + \gamma'_{32}\pi^{d_2}$$

$$(12) \quad \gamma_{31}\pi^{c_3} + \gamma_{32}\pi^{c_2+a_1} + g_{33}x\pi^{a_3} = g'_{11}y\pi^{b_3} + \gamma'_{21}\pi^{d_1+b_2} + \gamma'_{31}\pi^{d_3}.$$

We will assume first that c_1 and d_1 are not both zero and that c_2 and d_2 are not both zero. In this case we see that $\text{val}(\gamma'_{21}\pi^{d_1+b_2}) \geq \underline{d}_1 + b_2$, since if $d_1 = 0$ then $c_1 > 0$ forces $\text{val}(\gamma'_{21}) > 0$ by (10), and similarly $\text{val}(\gamma_{32}\pi^{c_2+a_1}) \geq \underline{c}_2 + a_1$.

If either of a_3 or b_3 is strictly less than $\min\{c_3, d_3, a_1 + \underline{c}_2, \underline{d}_1 + a_2\}$ then (12) implies that $a_3 = b_3$. However, a_3 and b_3 cannot be greater than this minimum, since $\mathbf{a}, \mathbf{b} \in \mathbf{T}_{\mathbf{c},\mathbf{d}}$, so the only other possibility is that they are both equal to it. Further, if either a_1 or b_1 is less than $\min\{\underline{c}_1, \underline{d}_1, a_3\}$ then (10) gives $a_1 = b_1$, but again the only other option is for them both to be equal to this minimum. Similarly, (11) shows that $a_2 = b_2$ and so we have $\mathbf{a} = \mathbf{b}$.

Now assume that $c_1 = d_1 = 0$ and note that this means that we may have γ'_{21} of valuation 0. In this case our triples \mathbf{a} and \mathbf{b} are such that $a_2 = a_3$ and $b_2 = b_3$ with $a_1 = b_1 = 1$. If either of a_2 or b_2 is less than $\min\{c_2, d_2\}$ then (11) implies that $a_2 = b_2$. Indeed, a_2 and b_2 cannot be greater than $\min\{c_2, d_2\}$ so we see that $\mathbf{a} = \mathbf{b}$. A similar argument deals with the case when $c_2 = d_2 = 0$. \square

Definition 5.3. For $\mathbf{a} \in \mathbf{T}_{\mathbf{c},\mathbf{d}}$ let

$$\mathbf{a}(\mathbf{c}, \mathbf{d}) = \min'\{a_1, a_2, a_3 - a_1, a_3 - a_2, c_i - a_i, d_i - a_i, a_1 + c_2 - a_3, d_1 + a_2 - a_3\}$$

and

$$\mathbf{a}(\mathbf{c}, \mathbf{d})' = \min'\{d_3 - a_3, c_3 - a_3, c_1 - a_1, d_2 - a_2\} \geq \mathbf{a}(\mathbf{c}, \mathbf{d})$$

where \min' means that we take 0 if any of the terms is negative.

Lemma 5.4. Let $\mathbf{c}, \mathbf{d} \in \mathbf{T}$, then

$$\mathbf{R}_{\mathbf{c},\mathbf{d}}^1 = \{t_{\mathbf{a},x} : \mathbf{a} \in \mathbf{T}_{\mathbf{c},\mathbf{d}}, x \in \mathcal{X}_{\mathbf{c},\mathbf{d}}^{\mathbf{a}}\}$$

where

$$\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}} = \begin{cases} (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})})^\times & \text{if } a_1 + a_2 \neq a_3; \\ \bigcup_{i=0}^{\mathbf{a}(\mathbf{c}, \mathbf{d})'} (1 + \pi^i \mathcal{R}^\times) \cap (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})+i})^\times \cap (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})'})^\times & \text{if } a_1 + a_2 = a_3. \end{cases}$$

Proof. Let g_{ij} , g'_{ij} , γ_{ij} and γ'_{ij} be as in the proof of Lemma 5.2, then $gt_{\mathbf{a}, x} = t_{\mathbf{a}, y}g'$ for some $x, y \in \mathcal{R}^\times$ precisely when the following three equations can be solved for g_{11} , g_{22} and g_{33} in \mathcal{R}^\times :

$$(13) \quad g_{11} = g_{22} - g_{12}\pi^{a_1} - g_{13}x\pi^{a_3} + \gamma_{21}\pi^{c_1-a_1} - \gamma'_{21}\pi^{d_1-a_1} + g_{23}x\pi^{a_3-a_1}$$

$$(14) \quad g_{33} = g_{22} - g_{12}r_y\pi^{-a_2} - g_{13}r_y + g_{23}\pi^{a_2} - \gamma_{32}\pi^{c_2-a_2} + \gamma'_{32}\pi^{d_2-a_2}$$

$$(x-y)g_{22} = (g_{12}\pi^{-a_2} - g_{23}\pi^{-a_1} + g_{13})xr_y + \gamma_{21}y\pi^{c_1-a_1} - \gamma_{31}\pi^{c_3-a_3}$$

$$(15) \quad -\gamma'_{32}x\pi^{d_2-a_2} + \gamma'_{31}\pi^{d_3-a_3} - \gamma_{32}r_x\pi^{c_2-a_2-a_3} + \gamma'_{21}r_y\pi^{d_1-a_1-a_3}$$

where for each $z \in \mathcal{R}^\times$ we define $r_z = \pi^{a_1+a_2} - z\pi^{a_3}$.

Suppose first that $a_1 + a_2 \neq a_3$, then (15) immediately yields

$$\text{val}(x-y) \geq \mathbf{a}(\mathbf{c}, \mathbf{d}).$$

Conversely, given distinct elements $x, y \in \mathcal{R}^\times$ with $\text{val}(x-y) \geq \mathbf{a}(\mathbf{c}, \mathbf{d})$ then one can solve (15) for $g_{22} \in \mathcal{R}^\times$ and a careful consideration of (13) and (14) reveals that one can choose these variables so that g_{11} and g_{33} are invertible as well. Thus the set

$$\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}} = (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})})^\times$$

exactly parametrises the representatives $t_{\mathbf{a}, x}$ for $\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}$ with $a_1 + a_2 \neq a_3$.

Now suppose that $a_1 + a_2 = a_3$. Let $\text{val}(1-x) = i$ and $\text{val}(1-y) = j$, then from (15) we see that

$$\text{val}(x-y) \geq \min\{a_1 + j, a_2 + j, c_2 - a_2 + i, d_1 - a_1 + j, \mathbf{a}(\mathbf{c}, \mathbf{d})'\}.$$

This clearly holds whenever $i, j \geq \mathbf{a}(\mathbf{c}, \mathbf{d})'$ since $\text{val}(x-y) \geq \min\{i, j\}$ so we will assume that at least one of i or j is less than $\mathbf{a}(\mathbf{c}, \mathbf{d})'$. If $i < j$ with $i < \mathbf{a}(\mathbf{c}, \mathbf{d})'$ then $\text{val}(x-y) = i$ and we must have $c_2 = a_2$. However, when $c_2 = a_2$ we see that (15) gives $\text{val}((x-y)g_{22} + \gamma_{32}(1-x)) > i$ which implies that $\text{val}(g_{22} - \gamma_{32}) > 0$, since if $\text{val}(g_{22} - \gamma_{32}) = 0$ then we would have

$$\text{val}((x-y)g_{22} + \gamma_{32}(1-x)) = \text{val}((1-y)g_{22} - (1-x)(g_{22} - \gamma_{32})) = i.$$

This in turn means that $\text{val}(g_{33}) = \text{val}(g_{22} - \gamma_{32}) > 0$ by (14) and so g_{33} cannot be invertible. Similarly, if $j < i$ with $j < \mathbf{a}(\mathbf{c}, \mathbf{d})'$ then $d_1 = a_1$ and g_{22} is not invertible by (13). Consequently, we must either have

$$(16) \quad \text{val}(1-x), \text{val}(1-y) \geq \mathbf{a}(\mathbf{c}, \mathbf{d})'$$

or

$$(17) \quad \begin{aligned} \text{val}(1-x) = \text{val}(1-y) = i < \mathbf{a}(\mathbf{c}, \mathbf{d})' \text{ and} \\ \text{val}(x-y) \geq \min\{\mathbf{a}(\mathbf{c}, \mathbf{d}) + i, \mathbf{a}(\mathbf{c}, \mathbf{d})'\}. \end{aligned}$$

Conversely, if $x, y \in \mathcal{R}^\times$ are distinct elements satisfying (16) or (17) then it is possible to find solutions to (13-15). Hence, the set

$$\bigcup_{i=0}^{\mathbf{a}(\mathbf{c}, \mathbf{d})'} (1 + \pi^i \mathcal{R}^\times) \cap (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})+i})^\times \cap (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})'})^\times$$

precisely parametrises the representatives $t_{\mathbf{a}, x}$ for $\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}$ with $a_1 + a_2 = a_3$. \square

Note that if $a_1 + a_2 = a_3$ then the set $\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}}$ lies between $(\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})})^\times$ and $(\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})'})^\times$. In particular, when $\mathbf{a}(\mathbf{c}, \mathbf{d})' = \mathbf{a}(\mathbf{c}, \mathbf{d})$ then the definition of $\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}}$ given in Lemma 5.4 reduces to the much simpler

$$\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}} = (\mathcal{R}/\mathcal{P}^{\mathbf{a}(\mathbf{c}, \mathbf{d})})^\times.$$

Further, in general we can compute directly that

$$|\mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}}| = \begin{cases} 1 & \text{if } a_1 + a_2 \neq a_3, \mathbf{a}(\mathbf{c}, \mathbf{d}) = 0; \\ (q-1)q^{\mathbf{a}(\mathbf{c}, \mathbf{d})-1} & \text{if } a_1 + a_2 \neq a_3, \mathbf{a}(\mathbf{c}, \mathbf{d}) > 0; \\ \mathbf{a}(\mathbf{c}, \mathbf{d})' + 1 & \text{if } a_1 + a_2 = a_3, \mathbf{a}(\mathbf{c}, \mathbf{d}) = 0; \\ (\mathbf{a}(\mathbf{c}, \mathbf{d})' - \mathbf{a}(\mathbf{c}, \mathbf{d}) + 1)(q-1)q^{\mathbf{a}(\mathbf{c}, \mathbf{d})-1} & \text{if } a_1 + a_2 = a_3, \mathbf{a}(\mathbf{c}, \mathbf{d}) > 0. \end{cases}$$

Theorem 5.5. *Let $\mathbf{c}, \mathbf{d} \in \mathbf{T}$, then a complete set of $(C_{\mathbf{c}}, C_{\mathbf{d}})$ -double coset representatives in K is*

$$\mathbf{R}_{\mathbf{c}, \mathbf{d}} = \bigcup_{w \in W_{\mathbf{c}, \mathbf{d}}} \mathbf{R}_{\mathbf{c}, \mathbf{d}}^w$$

where if $w \in W_{\mathbf{c}, \mathbf{d}}$ then we define $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^w$ as follows

- (i) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^1 = \{t_{\mathbf{a}, x} : \mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}, x \in \mathbf{X}_{\mathbf{c}, \mathbf{d}}^{\mathbf{a}}\};$
- (ii) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{s_1^{(\alpha, \beta)}} = \{s_1^{(\alpha, \beta)} : 1 \leq \alpha \leq \min\{\underline{d}_2, c_3\}, 1 \leq \beta \leq \min\{\underline{c}_2, d_3\}, -c_1 \leq \beta - \alpha \leq d_1\};$
- (iii) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{s_2^{(\alpha, \beta)}} = \{s_2^{(\alpha, \beta)} : 1 \leq \alpha \leq \min\{\underline{d}_1, c_3\}, 1 \leq \beta \leq \min\{\underline{c}_1, d_3\}, -c_2 \leq \beta - \alpha \leq d_2\};$
- (iv) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{s_1 s_2^{(\alpha)}} = \{s_1 s_2^{(\alpha)} : 1 \leq \alpha \leq \min\{d_1, c_2\}\};$
- (v) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{s_2 s_1^{(\alpha)}} = \{s_2 s_1^{(\alpha)} : 1 \leq \alpha \leq \min\{c_1, d_2\}\};$
- (vi) $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{w_0} = \{w_0\}$

and otherwise we take $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^w = \emptyset$.

Proof. We have already shown (i) in Lemma 5.4 so we need to consider the representatives corresponding to non-trivial Weyl group elements. As in Theorem 4.3 we will prove the case where $w = s_1$ and note that the other cases are similar.

Suppose that $s_1^{(\alpha, \beta)}$ and $s_1^{(\alpha', \beta')}$ represent the same double coset for pairs (α, β) and (α', β') with $\alpha, \alpha', \beta, \beta' \geq 1$. There must therefore be elements $g = [g_{ij}]$ of $C_{\mathbf{c}}$ and $g' = [g'_{ij}]$ of $C_{\mathbf{d}}$ such that $gs_1^{(\alpha, \beta)} = s_1^{(\alpha', \beta')}g'$ and, if we let γ_{ij} and γ'_{ij} be as in the proof of Lemma 5.2, this occurs precisely when we can find $g_{11}, g_{22}, g_{33} \in \mathcal{R}^\times$ and $g_{13}, g_{23}, \gamma_{21}, \gamma'_{21}, \gamma_{32}, \gamma'_{32}, \gamma_{31}, \gamma'_{31} \in \mathcal{R}$ with

$$(18) \pi^{\alpha'} g_{11} = \pi^{\alpha} g_{33} - \pi^{\alpha + \alpha'} g_{13} - \pi^{\alpha + \beta'} g_{23} - \pi^{c_1 + \beta'} \gamma_{21} - \pi^{d_2} \gamma'_{32} + \pi^{c_3} \gamma_{31}$$

$$(19) \pi^{\beta'} g_{22} = \pi^{\beta} g_{33} - \pi^{\beta + \beta'} g_{23} - \pi^{d_1 + \alpha'} \gamma'_{21} + \pi^{c_2} \gamma_{32} - \pi^{d_3} \gamma'_{31}.$$

First note that we must have $\text{val}(\pi^{d_2}\gamma'_{32}) \geq \underline{d}_2$ since if $d_2 = 0$ then (18) forces $\text{val}(\gamma'_{32}) \geq 1$ and, similarly, $\text{val}(\pi^{c_2}\gamma_{32}) \geq \underline{c}_2$ by (19). It follows that one can solve (18) and (19) whenever $\alpha' \geq \min\{\alpha, c_1 + \beta', \underline{d}_2, c_3\}$ and $\beta' \geq \min\{\beta, d_1 + \alpha', \underline{c}_2, d_3\}$. Thus we may choose a representative with (α, β) such that $1 \leq \alpha \leq \min\{\underline{d}_2, c_3\}$, $1 \leq \beta \leq \min\{\underline{c}_2, d_3\}$ and $-c_1 \leq \beta - \alpha \leq d_1$.

Now suppose that $s_1^{(\alpha, \beta)}$ and $s_1^{(\alpha', \beta')}$ are representatives for the same double coset where (α, β) and (α', β') satisfy the restrictions above. If either of α or α' is less than $\min\{c_1 + \beta', \underline{d}_2, c_3\}$ then $\alpha = \alpha'$ by (18). Further, if α and α' are greater than or equal to this minimum we actually have $\alpha \geq \alpha' = c_1 + \beta'$ with $\alpha = \alpha'$ whenever $\beta = \beta'$. Similarly, if at least one of β or β' is less than $\min\{d_1 + \alpha', \underline{c}_2, d_3\}$ then $\beta = \beta'$ by (19) and otherwise $\beta \geq \beta' = d_1 + \alpha'$ with $\beta = \beta'$ whenever $\alpha = \alpha'$. However, we cannot have both $\alpha' = c_1 + \beta'$ and $\beta' = d_1 + \alpha'$, since this would mean that $c_1 = d_1 = 0$, so we must have $\alpha = \alpha'$ and $\beta = \beta'$. Hence distinct pairs (α, β) give rise to distinct double cosets. \square

Remark 5.6. The list of double coset representatives $\mathbf{R}_{\mathbf{c}, \mathbf{d}}$ given in Theorem 5.5 does not seem to be symmetric in \mathbf{c} and \mathbf{d} . There is, however, a natural bijection from $C_{\mathbf{c}} \backslash K / C_{\mathbf{d}}$ to $C_{\mathbf{d}} \backslash K / C_{\mathbf{c}}$ obtained by sending each element of a double coset to its inverse. This does indeed induce a bijection from $\mathbf{R}_{\mathbf{c}, \mathbf{d}}$ to $\mathbf{R}_{\mathbf{d}, \mathbf{c}}$ since we see that

$$\begin{aligned} (C_{\mathbf{c}} t_{\mathbf{a}, x} C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} t_{\mathbf{b}, y} C_{\mathbf{c}} \\ (C_{\mathbf{c}} s_1^{(\alpha, \beta)} C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} s_1^{(\beta, \alpha)} C_{\mathbf{c}} \\ (C_{\mathbf{c}} s_2^{(\alpha, \beta)} C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} s_2^{(\beta, \alpha)} C_{\mathbf{c}} \\ (C_{\mathbf{c}} s_1 s_2^{(\alpha)} C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} s_2 s_1^{(\alpha)} C_{\mathbf{c}} \\ (C_{\mathbf{c}} s_2 s_1^{(\alpha)} C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} s_1 s_2^{(\alpha)} C_{\mathbf{c}} \\ (C_{\mathbf{c}} w_0 C_{\mathbf{d}})^{-1} &= C_{\mathbf{d}} w_0 C_{\mathbf{c}} \end{aligned}$$

where

$$(\mathbf{b}, y) = \begin{cases} (\mathbf{a}, x - \pi^{a_3 - a_1 - a_2}) & \text{if } a_1 + a_2 < a_3; \\ ((a_1, a_2, \text{val}(r_x)), r_x \pi^{-\text{val}(r_x)}) & \text{if } a_1 + a_2 = a_3; \\ ((a_1, a_2, a_1 + a_2), -1 + x \pi^{a_1 + a_2 - a_3}) & \text{if } a_1 + a_2 > a_3. \end{cases}$$

In particular, Theorem 5.5 is symmetric in \mathbf{c} and \mathbf{d} with respect to this bijection.

We want to use the description of the double coset structure given Theorem 5.5 to investigate the components $V_{\mathbf{c}}$. From Corollary 3.4 we know that

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = \sum_{I \subseteq S_{\mathbf{c}}, J \subseteq S_{\mathbf{d}}} (-1)^{|I|+|J|} |\mathbf{R}_{\mathbf{c}_I, \mathbf{d}_J}|$$

and for each $w \in W$ we will consider

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}})^w = \sum_{I \subseteq S_{\mathbf{c}}, J \subseteq S_{\mathbf{d}}} (-1)^{|I|+|J|} |\mathbf{R}_{\mathbf{c}_I, \mathbf{d}_J}^w|$$

since then $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = \sum_{w \in W} \mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}})^w$.

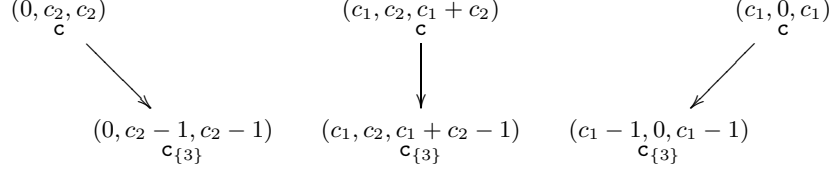


FIGURE 1. The one descendant cases

6. ONE DESCENDANT

Rather than consider all possible \mathbf{c} , we split the problem into three separate cases depending on the number of triples immediately below \mathbf{c} in the poset \mathbf{T} . Recall that the space of K_1 -fixed vectors in V decomposes as

$$V^{K_1} = V_{(0,0,0)} \oplus V_{(0,1,1)} \oplus V_{(1,0,1)} \oplus V_{(1,1,1)}$$

where $(0, 1, 1)$ and $(1, 0, 1)$ are the triples having exactly one descendant in \mathbf{T} . The corresponding components $V_{(0,1,1)}$ and $V_{(1,0,1)}$ are equivalent irreducibles and we find that single descendant triples (see Figure 1) will always give irreducible components which are equivalent to all other single descendant components lying in the same level.

Theorem 6.1. *Let $\mathbf{c} = (c_1, c_2, c_1 + c_2)$ with $c_1 + c_2 > 1$, then $V_{\mathbf{c}}$ is irreducible and*

$$\dim V_{\mathbf{c}} = q^{2c_1+2c_2-4}(q-1)(q+1)(q^2+q+1).$$

Moreover, $V_{\mathbf{c}} \simeq V_{\mathbf{d}}$ for any $\mathbf{d} = (d_1, d_2, d_1 + d_2)$ with $c_1 + c_2 = d_1 + d_2$.

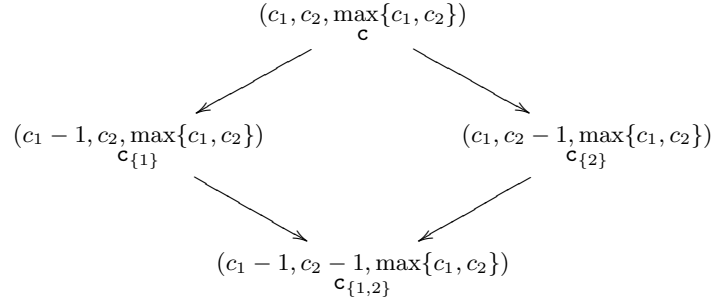
Proof. For ease of notation we will assume that $c_1 \leq d_1$. If we define $\mathbf{c}' = \mathbf{c}_{\{3\}}$ and $\mathbf{d}' = \mathbf{d}_{\{3\}}$ then we want to calculate the alternating sum

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = |\mathbf{R}_{\mathbf{c}, \mathbf{d}}| - |\mathbf{R}_{\mathbf{c}', \mathbf{d}}| - |\mathbf{R}_{\mathbf{c}, \mathbf{d}'}| + |\mathbf{R}_{\mathbf{c}', \mathbf{d}'}|.$$

First note that if c_1, d_1, c_2 and d_2 are all non-zero then $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^w, \mathbf{R}_{\mathbf{c}', \mathbf{d}}^w, \mathbf{R}_{\mathbf{c}, \mathbf{d}'}^w$ and $\mathbf{R}_{\mathbf{c}', \mathbf{d}'}^w$ are all equal for any non-trivial Weyl group element $w \in W$ since we are only decreasing c_3 or d_3 in Theorem 5.5 and these are both greater than $\max\{c_1, d_1, c_2, d_2\}$. In the case where one or more of c_1, d_1, c_2 or d_2 is zero the sets are equal in pairs since $c_3 = d_3 > 1$. Consequently, we need only consider

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}})^1 = |\mathbf{R}_{\mathbf{c}, \mathbf{d}}^1| - |\mathbf{R}_{\mathbf{c}', \mathbf{d}}^1| - |\mathbf{R}_{\mathbf{c}, \mathbf{d}'}^1| + |\mathbf{R}_{\mathbf{c}', \mathbf{d}'}^1|.$$

Now, from Definition 5.3 we see that $\mathbf{T}_{\mathbf{c}', \mathbf{d}}, \mathbf{T}_{\mathbf{c}, \mathbf{d}'}$ and $\mathbf{T}_{\mathbf{c}', \mathbf{d}'}$ are equal while $\mathbf{T}_{\mathbf{c}, \mathbf{d}}$ contains the additional triple $\mathbf{a} = (\underline{c}_1, \underline{d}_2, c_1 + c_2)$ with $\mathbf{a}(\mathbf{c}, \mathbf{d}) = 0$. If one of c_1, d_1, c_2 or d_2 is zero then $\mathbf{a}(\mathbf{c}, \mathbf{d}), \mathbf{a}(\mathbf{c}', \mathbf{d}), \mathbf{a}(\mathbf{c}, \mathbf{d}')$ and $\mathbf{a}(\mathbf{c}', \mathbf{d}')$ are all zero for every $\mathbf{a} \in \mathbf{T}_{\mathbf{c}', \mathbf{d}'}$. On the other hand, if c_1, d_1, c_2 and d_2 are all non-zero then the only way that a triple $\mathbf{a} \in \mathbf{T}_{\mathbf{c}', \mathbf{d}'}$ could have $\mathbf{a}(\mathbf{c}, \mathbf{d})$ strictly greater than any of $\mathbf{a}(\mathbf{c}', \mathbf{d}), \mathbf{a}(\mathbf{c}, \mathbf{d}')$ or $\mathbf{a}(\mathbf{c}', \mathbf{d}')$ is if the minimum occurs for $c_3 - a_3$. However, $c_3 - a_3 = c_1 + c_2 - a_3 \geq a_1 + c_2 - a_3$ so we would need $a_1 = c_1$ and the minimum would therefore have been 0. Thus again $\mathbf{a}(\mathbf{c}, \mathbf{d}), \mathbf{a}(\mathbf{c}', \mathbf{d}), \mathbf{a}(\mathbf{c}, \mathbf{d}')$ and $\mathbf{a}(\mathbf{c}', \mathbf{d}')$

FIGURE 2. The general two descendant case with $c_1, c_2 > 1$

must be equal for every $\mathbf{a} \in \mathbf{T}_{\mathbf{c}', \mathbf{d}'}$. This implies that $\mathbf{R}_{\mathbf{c}', \mathbf{d}}^1$, $\mathbf{R}_{\mathbf{c}, \mathbf{d}'}^1$ and $\mathbf{R}_{\mathbf{c}', \mathbf{d}'}^1$ are equal while $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^1$ has the extra representative $t_{(\underline{c}_1, \underline{d}_2, c_1 + c_2), 1}$. Hence

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = (|\mathbf{R}_{\mathbf{c}', \mathbf{d}}^1| + 1) - |\mathbf{R}_{\mathbf{c}', \mathbf{d}'}^1| - |\mathbf{R}_{\mathbf{c}, \mathbf{d}'}^1| + |\mathbf{R}_{\mathbf{c}', \mathbf{d}'}^1| = 1.$$

Taking $\mathbf{c} = \mathbf{d}$ this shows $V_{\mathbf{c}}$ is irreducible and in general it implies that $V_{\mathbf{c}}$ and $V_{\mathbf{d}}$ must be equivalent. \square

In fact, the equivalences in Theorem 6.1 are the only ones that can involve a component $V_{\mathbf{c}}$ with a triple of the form $\mathbf{c} = (c_1, c_2, c_1 + c_2)$.

Proposition 6.2. *Let $\mathbf{c} = (c_1, c_2, c_1 + c_2)$ with $c_1 + c_2 > 1$, then the multiplicity of $V_{\mathbf{c}}$ in $\text{Res}_K^{\mathbb{G}(F)} V$ is $c_1 + c_2 + 1$.*

Proof. By Theorem 6.1 we may take $\mathbf{c} = (0, c, c)$ where $c = c_1 + c_2$. Further, since K_c is the largest principal congruence subgroup contained in $C_{(0, c, c)}$, we know that every subrepresentation of $\text{Res}_K^{\mathbb{G}(F)} V$ equivalent to $V_{\mathbf{c}}$ must be a subrepresentation of $V^{K_c} \simeq U_{(c, c, c)}$. In particular, this means that the multiplicity of $V_{\mathbf{c}}$ in $\text{Res}_K^{\mathbb{G}(F)} V$ is equal to its multiplicity in $U_{(c, c, c)}$. Thus, setting $\mathbf{c}' = \mathbf{c}_{\{3\}} = (0, c - 1, c - 1)$ and $\mathbf{d} = (c, c, c)$ we would like to calculate

$$\mathcal{J}(V_{\mathbf{c}}, U_{\mathbf{d}}) = |\mathbf{R}_{\mathbf{c}, \mathbf{d}}| - |\mathbf{R}_{\mathbf{c}', \mathbf{d}}|.$$

Now, $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{w_0} = \mathbf{R}_{\mathbf{c}', \mathbf{d}}^{w_0} = \{w_0\}$ and $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^w = \mathbf{R}_{\mathbf{c}', \mathbf{d}}^w = \emptyset$ for $w = s_1, s_1 s_2$ or $s_2 s_1$. However, for $w = s_2$ we obtain $\mathbf{R}_{\mathbf{c}, \mathbf{d}}^{s_2} = \mathbf{R}_{\mathbf{c}', \mathbf{d}}^{s_2} \cup \{s_2^{(c, 1)}\}$. Further, for every $\mathbf{a} \in \mathbf{T}_{\mathbf{c}', \mathbf{d}}$ we have $\mathbf{a}(\mathbf{c}, \mathbf{d}) = \mathbf{a}(\mathbf{c}', \mathbf{d}) = 0$ and $\mathbf{T}_{\mathbf{c}, \mathbf{d}} = \mathbf{T}_{\mathbf{c}', \mathbf{d}} \cup \{(1, a, c) : 1 \leq a \leq c\}$. Hence $\mathbf{R}_{\mathbf{c}, \mathbf{d}} = \mathbf{R}_{\mathbf{c}', \mathbf{d}} \cup \{t_{(1, a, c), 1} : 1 \leq a \leq c\} \cup \{s_2^{(c, 1)}\}$ and

$$\mathcal{J}(V_{\mathbf{c}}, U_{\mathbf{d}}) = (|\mathbf{R}_{\mathbf{c}', \mathbf{d}}| + (c + 1)) - |\mathbf{R}_{\mathbf{c}', \mathbf{d}}| = c + 1$$

as required. \square

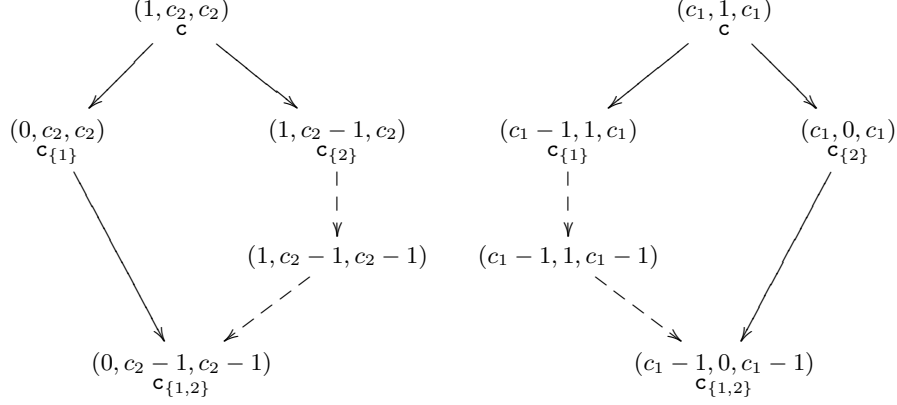


FIGURE 3. The extremal two descendant cases

7. TWO DESCENDANTS

In the decomposition of V^{K_1} the only component corresponding to a triple with exactly two descendants in \mathbf{T} is $V_{(1,1,1)}$. This is the pull-back to K of the Steinberg representation of $\mathrm{GL}(3, \mathbf{f})$ so is irreducible and appears with multiplicity 1. Indeed, any triple $\mathbf{c} \in \mathbf{T}$ with two descendants (see Figure 2) will give an irreducible component $V_{\mathbf{c}}$ which has multiplicity 1 in the restriction of V to K . Here we note that if $c_1 = 1$ then $\mathbf{c}_{\{2\}} = (1, c_2 - 1, c_2)$ but $\mathbf{c}_{\{1,2\}} = (0, c_2 - 1, c_2 - 1)$ so for the purposes of calculating $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}})$ we ignore the triple in \mathbf{T} that lies between them (see Figure 3). Similarly, we ignore the triple between $\mathbf{c}_{\{1\}}$ and $\mathbf{c}_{\{1,2\}}$ when $c_2 = 1$.

Theorem 7.1. *Let $\mathbf{c} = (c_1, c_2, \max\{c_1, c_2\})$ where $c_1, c_2 \geq 1$ and $c_1 + c_2 > 1$, then $V_{\mathbf{c}}$ is irreducible of dimension*

$$\dim V_{\mathbf{c}} = q^{c_1 + c_2 + \max\{c_1, c_2\} - 5} (q - 1)^2 (q + 1) (q^2 + q + 1).$$

Moreover, the multiplicity of $V_{\mathbf{c}}$ in $\mathrm{Res}_K^{\mathbb{G}(F)} V$ is 1.

Proof. We will assume that $c_1 \leq c_2$ so that $\mathbf{c} = (c_1, c_2, c_2)$ and remark that the proof for the case where $c_1 \geq c_2$ is similar. Let $\mathbf{d} = (c_2, c_2, c_2)$, then as in Proposition 6.2 it suffices to calculate

$$\mathcal{J}(V_{\mathbf{c}}, U_{\mathbf{d}}) = |\mathbf{R}_{\mathbf{c}_{\emptyset}, \mathbf{d}}| - |\mathbf{R}_{\mathbf{c}_{\{1\}}, \mathbf{d}}| - |\mathbf{R}_{\mathbf{c}_{\{2\}}, \mathbf{d}}| + |\mathbf{R}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}|.$$

First suppose that $c_1 > 1$. We begin by examining the double cosets corresponding to non-trivial $w \in W$. Let $s_1^{(\alpha, \beta)}$ be a representative which belongs to $\mathbf{R}_{\mathbf{c}_{\emptyset}, \mathbf{d}}^{s_1}$ but not to $\mathbf{R}_{\mathbf{c}_{\{2\}}, \mathbf{d}}$. Then $\beta = c_2$, since we are decreasing c_2 , and $1 \leq \alpha \leq c_2$, since the restriction $-c_1 \leq c_2 - \alpha \leq c_2$ does not play a role. Thus we see that $\mathbf{R}_{\mathbf{c}_{\emptyset}, \mathbf{d}}^{s_1} = \mathbf{R}_{\mathbf{c}_{\{2\}}, \mathbf{d}}^{s_1} \cup \{s_1^{(\alpha, c_2)} : 1 \leq \alpha \leq c_2\}$ and $\mathbf{R}_{\mathbf{c}_{\{1\}}, \mathbf{d}}^{s_1} = \mathbf{R}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}^{s_1} \cup \{s_1^{(\alpha, c_2)} : 1 \leq \alpha \leq c_2\}$ by the same argument. This means

that the contribution of these representatives to the alternating sum is

$$\mathcal{J}(V_c, U_d)^{s_1} = (|\mathbf{R}_{c_{\{2\},d}}^{s_1}| + c_2) - (|\mathbf{R}_{c_{\{1,2\},d}}^{s_1}| + c_2) - |\mathbf{R}_{c_{\{2\},d}}^{s_1}| + |\mathbf{R}_{c_{\{1,2\},d}}^{s_1}| = 0.$$

Similarly, let $s_2^{(\alpha,\beta)}$ be a representative lying in $\mathbf{R}_{c_{\emptyset},d}^{s_2}$ but not in $\mathbf{R}_{c_{\{1\},d}}^{s_2}$. We are now decreasing c_1 so $\beta = c_1$ and $1 \leq \alpha \leq c_2$ since the restriction $-c_2 \leq c_1 - \alpha \leq c_2$ is again irrelevant. We therefore obtain $\mathbf{R}_{c_{\emptyset},d}^{s_2} = \mathbf{R}_{c_{\{1\},d}}^{s_2} \cup \{s_2^{(\alpha,c_1)} : 1 \leq \alpha \leq c_2\}$ and $\mathbf{R}_{c_{\{2\},d}}^{s_2} = \mathbf{R}_{c_{\{1,2\},d}}^{s_2} \cup \{s_2^{(\alpha,c_1)} : 1 \leq \alpha \leq c_2\}$ in the same manner giving

$$\mathcal{J}(V_c, U_d)^{s_2} = (|\mathbf{R}_{c_{\{1\},d}}^{s_2}| + c_2) - |\mathbf{R}_{c_{\{1\},d}}^{s_2}| - (|\mathbf{R}_{c_{\{1,2\},d}}^{s_2}| + c_2) + |\mathbf{R}_{c_{\{1,2\},d}}^{s_2}| = 0.$$

Further, it is easy to check that we also have $\mathcal{J}(V_c, U_d)^w = 0$ for $w = s_1 s_2$, $s_2 s_1$ and w_0 . Hence, we only need to consider

$$\mathcal{J}(V_c, U_d)^1 = |\mathbf{R}_{c_{\emptyset},d}^1| - |\mathbf{R}_{c_{\{1\},d}}^1| - |\mathbf{R}_{c_{\{2\},d}}^1| + |\mathbf{R}_{c_{\{1,2\},d}}^1|.$$

Now, $\mathbf{T}_{c_{\emptyset},d} = \mathbf{T}_{c_{\{2\},d} \cup \{(a, c_2, c_2) : 1 \leq a \leq c_1\}}$ and for each $\mathbf{a} \in \mathbf{T}_{c_{\{2\},d}}$ the only way that we can have $\mathbf{a}(c_{\emptyset}, d) > \mathbf{a}(c_{\{2\},d})$ is if the minimum occurs for $c_2 - a_2$. However, since $c_2 - a_2 = c_3 - a_2 \geq c_3 - a_3$, this would imply that $a_3 = a_2$ and so we would in fact have $\mathbf{a}(c_{\emptyset}, d) = \mathbf{a}(c_{\{2\},d}) = 0$. Consequently $\mathbf{a}(c_{\emptyset}, d) = \mathbf{a}(c_{\{2\},d})$ for every $\mathbf{a} \in \mathbf{T}_{c_{\{2\},d}}$ and $\mathbf{R}_{c_{\emptyset},d}^1 = \mathbf{R}_{c_{\{2\},d}}^1 \cup \{t_{(a,c_2,c_2),1} : 1 \leq a \leq c_1\}$. Similarly, $\mathbf{T}_{c_{\{1\},d} = \mathbf{T}_{c_{\{1,2\},d} \cup \{(a, c_2, c_2) : 1 \leq a \leq c_1 - 1\}}$ and $\mathbf{a}(c_{\{1\},d}) = \mathbf{a}(c_{\{1,2\},d})$ for every $\mathbf{a} \in \mathbf{T}_{c_{\{1,2\},d}}$, implying that $\mathbf{R}_{c_{\{1\},d}}^1 = \mathbf{R}_{c_{\{1,2\},d}}^1 \cup \{t_{(a,c_2,c_2),1} : 1 \leq a \leq c_1 - 1\}$. Hence we obtain

$$\mathcal{J}(V_c, U_d)^1 = (|\mathbf{R}_{c_{\{2\},d}}^1| + c_1) - (|\mathbf{R}_{c_{\{1,2\},d}}^1| + (c_1 - 1)) - |\mathbf{R}_{c_{\{2\},d}}^1| + |\mathbf{R}_{c_{\{1,2\},d}}^1| = 1.$$

and V_c is an irreducible subrepresentation of U_d with multiplicity 1.

Suppose now that $c_1 = 1$ so that $\mathbf{c} = (1, c_2, c_2)$ and $c_{\{1,2\}} = (0, c_2 - 1, c_2 - 1)$. While we still have $\mathbf{R}_{c_{\emptyset},d}^{s_1} = \mathbf{R}_{c_{\{2\},d}}^{s_1} \cup \{s_1^{(\alpha,c_2)} : 1 \leq \alpha \leq c_2\}$ it transpires that $\mathbf{R}_{c_{\{1\},d}}^{s_1} = \mathbf{R}_{c_{\{1,2\},d}}^{s_1} = \emptyset$ since s_1 does not belong to $W_{c_{\{1\},d}}$ or $W_{c_{\{1,2\},d}}$. Thus in this case the contribution from these representatives becomes

$$\mathcal{J}(V_c, U_d)^{s_1} = (|\mathbf{R}_{c_{\{2\},d}}^{s_1}| + c_2) - 0 - |\mathbf{R}_{c_{\{2\},d}}^{s_1}| + 0 = c_2.$$

In contrast, reducing c_1 no longer changes the inequalities in Theorem 5.5(iii) so $\mathbf{R}_{c_{\emptyset},d}^{s_2} = \mathbf{R}_{c_{\{1\},d}}^{s_2}$ and $\mathbf{R}_{c_{\{2\},d}}^{s_2} = \mathbf{R}_{c_{\{1,2\},d}}^{s_2} \cup \{s_2^{(\alpha,c_2)}\}$ since in $c_{\{1,2\}}$ we also decrease the third entry. Consequently,

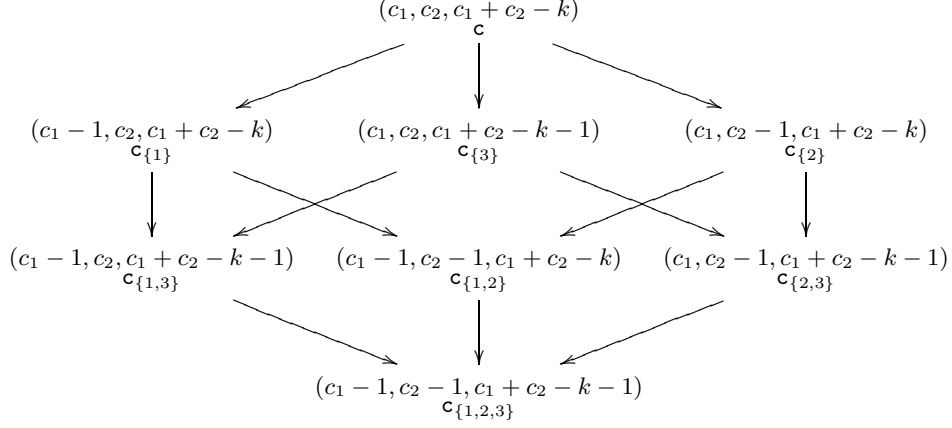
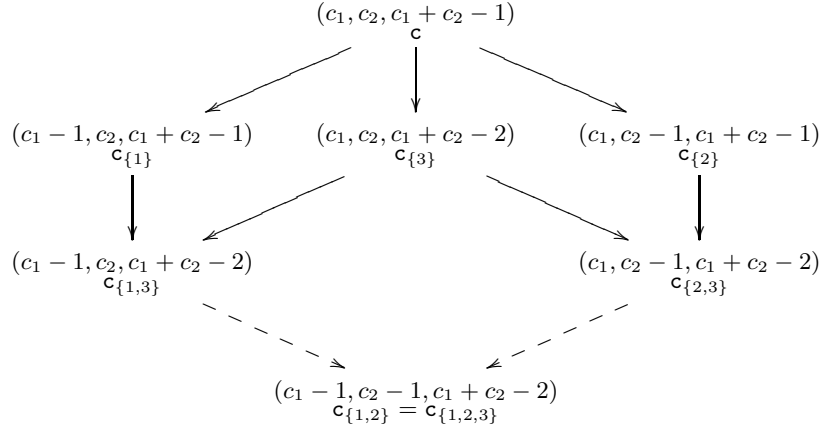
$$\mathcal{J}(V_c, U_d)^{s_2} = |\mathbf{R}_{c_{\{1\},d}}^{s_2}| - |\mathbf{R}_{c_{\{1\},d}}^{s_2}| - (|\mathbf{R}_{c_{\{1,2\},d}}^{s_2}| + 1) + |\mathbf{R}_{c_{\{1,2\},d}}^{s_2}| = -1.$$

Further, $\mathbf{R}_{c_{\emptyset},d}^{s_1 s_2} = \mathbf{R}_{c_{\{2\},d}}^{s_1 s_2} \cup \{s_1 s_2^{(c_2)}\}$ and $\mathbf{R}_{c_{\{1\},d}}^{s_1 s_2} = \mathbf{R}_{c_{\{1,2\},d}}^{s_1 s_2} = \emptyset$ giving

$$\mathcal{J}(V_c, U_d)^{s_1 s_2} = (|\mathbf{R}_{c_{\{2\},d}}^{s_1 s_2}| + 1) - 0 - |\mathbf{R}_{c_{\{2\},d}}^{s_1 s_2}| + 0 = 1$$

while $\mathcal{J}(V_c, U_d)^w = 0$ for $w = s_2 s_1$ and w_0 .

Now, $\mathbf{T}_{c_{\emptyset},d} = \mathbf{T}_{c_{\{1\},d}$ with $\mathbf{a}(c_{\emptyset}, d) = \mathbf{a}(c_{\{1\},d}) = 0$ for all $\mathbf{a} \in \mathbf{T}_{c_{\{1\},d}$ since we will always have $c_1 = a_1$. Thus $\mathbf{R}_{c_{\emptyset},d}^1$ and $\mathbf{R}_{c_{\{1\},d}}^1$ are equal. However, since in $c_{\{1,2\}}$ we also reduce the c_3 entry, we have $\mathbf{T}_{c_{\{2\},d} = \mathbf{T}_{c_{\{1,2\},d} \cup \{(1, a_2, c_2) :$

FIGURE 4. The three descendant case with $1 < k < \min\{c_1, c_2\}$ FIGURE 5. The three descendant case with $k = 1$

$1 \leq a_2 \leq c_2 - 1\}$. Again $\mathbf{a}(\mathbf{c}_{\{2\}}, \mathbf{d}) = \mathbf{a}(\mathbf{c}_{\{1,2\}}, \mathbf{d}) = 0$ for every $\mathbf{a} \in \mathbf{T}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}$ so $\mathbf{R}_{\mathbf{c}_{\{1\}}, \mathbf{d}}^1 = \mathbf{R}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}^1 \cup \{t_{(1, a_2, c_2), 1} : 1 \leq a_2 \leq c_2 - 1\}$. Thus

$$\mathcal{J}(V_{\mathbf{c}}, U_{\mathbf{d}})^1 = |\mathbf{R}_{\mathbf{c}_{\{1\}}, \mathbf{d}}^1| - |\mathbf{R}_{\mathbf{c}_{\{1\}}, \mathbf{d}}^1| - (|\mathbf{R}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}^1| + (c_2 - 1)) + |\mathbf{R}_{\mathbf{c}_{\{1,2\}}, \mathbf{d}}^1| = -(c_2 - 1).$$

Hence, overall we obtain

$$\mathcal{J}(V_{\mathbf{c}}, U_{\mathbf{d}}) = -(c_2 - 1) + c_2 + (-1) + 1 + 0 + 0 = 1$$

and $V_{\mathbf{c}}$ is an irreducible subrepresentation of $U_{\mathbf{d}}$ with multiplicity 1. \square

8. THREE DESCENDANTS

The remaining case, where \mathbf{c} has three triples immediately beneath it in \mathbf{T} (see Figure 4), does not appear in the decomposition of V^{K_1} and we find that these components are reducible in general. Consider $\mathbf{c} = (c_1, c_2, c_3)$ as part of a chain of triples

$$(c_1, c_2, c_1 + c_2) \succeq \cdots \succeq \mathbf{c} \succeq \cdots \succeq (c_1, c_2, \max\{c_1, c_2\}).$$

We let k denote the position of \mathbf{c} in this chain, so that $c_3 = c_1 + c_2 - k$, and $\ell = \min\{c_1, c_2\}$ the length of the chain. The number of intertwining operators $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}})$ turns out to be a polynomial in q whose degree is the minimum of k and $\ell - k$. Further, two triples correspond to equivalent components precisely when their chains start at the same level $c_1 + c_2$, they have the same position k in their chain and that position is in the first half of the chain. Here we note that when $k = 1$ the triples $\mathbf{c}_{\{1,3\}}$ and $\mathbf{c}_{\{1,2,3\}}$ will be equal (see Figure 5) so their contributions will cancel in the alternating sum for $V_{\mathbf{c}}$.

Theorem 8.1. *Let $\mathbf{c} = (c_1, c_2, c_1 + c_2 - k)$ with $0 < k < \ell = \min\{c_1, c_2\}$, then*

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) = \begin{cases} q - 2 & \text{if } k = 1; \\ (q - 1)^2 q^{k-2} & \text{if } 1 < k \leq \lfloor \ell/2 \rfloor; \\ (q - 1) q^{\ell-k-1} & \text{if } \lfloor \ell/2 \rfloor < k < \ell - 1; \\ (q - 1) & \text{if } k = \ell - 1. \end{cases}$$

Moreover, let $\mathbf{d} = (d_1, d_2, d_1 + d_2 - k')$ with $0 < k' < \ell' = \min\{d_1, d_2\}$. If we have

- (i) $c_3 = d_3$;
- (ii) $k = k'$; and
- (iii) $k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor$

then $V_{\mathbf{c}} \simeq V_{\mathbf{d}}$, otherwise $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = 0$.

We will prove Theorem 8.1 in a series of steps. Let \mathbf{c} and \mathbf{d} be as above. When $c_3 \neq d_3$ it is clear that we will have $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = 0$ so we can assume that $c_3 = d_3$. In particular, this means that $d_1, d_2 < c_3$ and $c_1, c_2 < d_3$ so if $w \neq 1$ then we have $\mathbf{R}_{\mathbf{c}_I, \mathbf{d}_J}^w = \mathbf{R}_{\mathbf{c}_I, \mathbf{d}_{J \cup \{3\}}}^w$ for each $I, J \subseteq S = \{1, 2, 3\}$. The contribution from the double cosets supported on non-trivial Weyl group elements is therefore 0 and

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = \sum_{I, J \subseteq S} (-1)^{|I|+|J|} |\mathbf{R}_{\mathbf{c}_I, \mathbf{d}_J}^1|.$$

Consequently, for a fixed triple $\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}$ we will consider the alternating sum

$$\mathcal{J}_{\mathbf{a}} = \sum_{I, J \subseteq S} (-1)^{|I|+|J|} |\mathbf{x}_{\mathbf{c}_I, \mathbf{d}_J}^{\mathbf{a}}|$$

where we take $|\mathbf{x}_{\mathbf{c}_I, \mathbf{d}_J}^{\mathbf{a}}| = 0$ for $\mathbf{a} \notin \mathbf{T}_{\mathbf{c}_I, \mathbf{d}_J}$. This gives $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = \sum_{\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}} \mathcal{J}_{\mathbf{a}}$.

We begin by showing that $V_{\mathbf{c}}$ and $V_{\mathbf{d}}$ will have no constituents in common if conditions (i–iii) in the Theorem are not met.

Lemma 8.2. *If $k \neq k'$, then $\mathcal{J}_{\mathbf{a}} = 0$ for every $\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{d}}$.*

Proof. Assume to the contrary that \mathbf{a} is a triple in $T_{\mathbf{c},\mathbf{d}}$ with $J_{\mathbf{a}} \neq 0$. We begin by showing that this cannot happen in the case where $a_1 + a_2 \neq a_3$.

Suppose that $c_3 - a_3 > \mathbf{a}(\mathbf{c}, \mathbf{d})$. For each $I, J \subseteq S$ we see that \mathbf{a} belongs to $T_{\mathbf{c}_I, \mathbf{d}_J}$ precisely when it belongs to $T_{\mathbf{c}_{I \cup \{3\}}, \mathbf{d}_J}$. Moreover, if $\mathbf{a} \in T_{\mathbf{c}_I, \mathbf{d}_J}$ then $\mathbf{a}(\mathbf{c}_I, \mathbf{d}_J) = \mathbf{a}(\mathbf{c}_{I \cup \{3\}}, \mathbf{d}_J)$ since decreasing c_3 by 1 does not change the minimum. However, this means that $\mathbf{x}_{\mathbf{c}_I, \mathbf{d}_J}^{\mathbf{a}} = \mathbf{x}_{\mathbf{c}_{I \cup \{3\}}, \mathbf{d}_J}^{\mathbf{a}}$ and we actually have

$$J_{\mathbf{a}} = \sum_{J \subseteq S} \sum_{I \subseteq \{1,2\}} (-1)^{|I|+|J|} \left(|\mathbf{x}_{\mathbf{c}_I, \mathbf{d}_J}^{\mathbf{a}}| - |\mathbf{x}_{\mathbf{c}_{I \cup \{3\}}, \mathbf{d}_J}^{\mathbf{a}}| \right) = 0.$$

Now suppose that $c_3 - a_3 = \mathbf{a}(\mathbf{c}, \mathbf{d})$ but that $c_1 - a_1 > \mathbf{a}(\mathbf{c}, \mathbf{d})$. When $k > 1$ the same approach can be used to show that $J_{\mathbf{a}} = 0$ so we need only consider the $k = 1$ case. Then $c_3 = c_1 + c_2 - 1$ gives $a_1 + c_2 - a_3 = (c_3 - a_3) - (c_1 - a_1) + 1 < 1$ implying that $\mathbf{a}(\mathbf{c}, \mathbf{d}) = a_1 + c_2 - a_3 = 0$. In particular, $\mathbf{a} \notin T_{\mathbf{c}_I, \mathbf{d}_J}$ if $2 \in I$ and $\mathbf{a}(\mathbf{c}_I, \mathbf{d}_J) = \mathbf{a}(\mathbf{c}_{I \cup \{1\}}, \mathbf{d}_J)$ otherwise. This again means that

$$J_{\mathbf{a}} = \sum_{J \subseteq S} \sum_{I \subseteq \{3\}} (-1)^{|I|+|J|} \left(|\mathbf{x}_{\mathbf{c}_I, \mathbf{d}_J}^{\mathbf{a}}| - |\mathbf{x}_{\mathbf{c}_{I \cup \{1\}}, \mathbf{d}_J}^{\mathbf{a}}| \right) = 0.$$

Similarly, $J_{\mathbf{a}} = 0$ if $d_2 - a_2 > \mathbf{a}(\mathbf{c}, \mathbf{d})$ so the only triples \mathbf{a} that could correspond to non-zero $J_{\mathbf{a}}$ are those with $c_1 - a_1 = d_2 - a_2 = c_3 - a_3 = \mathbf{a}(\mathbf{c}, \mathbf{d})$. Note that in this case $a_1 + c_2 - a_3 = k$ and $d_1 + a_2 - a_3 = k'$ with $c_1 \leq d_1$ and $c_2 \geq d_2$. If $k < k'$ then we must have $c_1 < d_1$. Consequently, $d_1 - a_1$ and $d_1 + a_2 - a_3$ are both greater than $\mathbf{a}(\mathbf{c}, \mathbf{d})$ and we can show that $J_{\mathbf{a}} = 0$ since $k' > 1$. On the other hand, if $k > k'$ then $c_2 > d_2$ implies that $c_2 - a_2$ and $a_1 + c_2 - a_3$ are greater than $\mathbf{a}(\mathbf{c}, \mathbf{d})$ and $J_{\mathbf{a}} = 0$. Hence, when $k \neq k'$ there cannot be a triple \mathbf{a} with $a_1 + a_2 \neq a_3$ which has non-zero $J_{\mathbf{a}}$.

We now consider the case where $a_1 + a_2 = a_3$ and note that $\mathbf{a}(\mathbf{c}, \mathbf{d})$ reduces to the minimum of $a_1, a_2, c_2 - a_2, d_1 - a_1$ and $\mathbf{a}(\mathbf{c}, \mathbf{d})'$. By an argument essentially identical to that given above we see that $J_{\mathbf{a}}$ can only be non-zero for triples \mathbf{a} with $c_1 - a_1 = d_2 - a_2 = c_3 - a_3 = \mathbf{a}(\mathbf{c}, \mathbf{d})'$ and these have $c_2 - a_2 = k$ and $d_1 - a_1 = k'$. If $k < k'$ then $d_1 - a_1 > \mathbf{a}(\mathbf{c}, \mathbf{d})$ implying that $J_{\mathbf{a}} = 0$ whereas $k > k'$ gives $c_2 - a_2 > \mathbf{a}(\mathbf{c}, \mathbf{d})$ and again $J_{\mathbf{a}} = 0$. Hence, we again see that when $k \neq k'$ no triples \mathbf{a} with $a_1 + a_2 = a_3$ have non-zero $J_{\mathbf{a}}$. \square

Lemma 8.3. *If $\mathbf{c} \neq \mathbf{d}$ but $k = k'$, then*

$$J(V_{\mathbf{c}}, V_{\mathbf{d}}) = \begin{cases} J_{(c_1-k, d_2-k, c_3-k)} & \text{if } c_1 < d_1 \text{ and } k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor; \\ J_{(d_1-k, c_2-k, d_1+c_2-2k)} & \text{if } c_1 > d_1 \text{ and } k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $\mathbf{a} \in T_{\mathbf{c}, \mathbf{d}}$ has $J_{\mathbf{a}} \neq 0$. When $a_1 + a_2 \neq a_3$ the proof of Lemma 8.2 tells us that $\mathbf{a} = (c_1 - i, d_2 - i, d_3 - i)$ where $i = \mathbf{a}(\mathbf{c}, \mathbf{d})$ and, moreover, that this can only happen if $c_1 < d_1$ and $c_2 > d_2$. If $k > \mathbf{a}(\mathbf{c}, \mathbf{d})$ then $d_1 - a_1$ and $d_1 + a_2 - a_3 = k$ are both greater than $\mathbf{a}(\mathbf{c}, \mathbf{d})$ which implies that $J_{\mathbf{a}} = 0$. Thus we must have $i = k$ and $k \leq \min\{a_1, a_2, a_3 - a_1, a_3 - a_2\} = \min\{\ell, \ell'\} - k$ gives $k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor$.

Similarly, when $a_1 + a_2 = a_3$ we know that $\mathbf{a} = (d_1 - k, c_2 - k, d_1 + c_2 - 2k)$ where $k = \mathbf{a}(\mathbf{c}, \mathbf{d})$ and that this only happens for $c_1 > d_1$ and $c_2 < d_2$. Further, $k \leq \min\{a_1, a_2\} = \min\{\ell, \ell'\} - k$ again implies that $k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor$. \square

Lemma 8.4. *For any \mathbf{c} ,*

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) = \sum_{i=0}^{\min\{k, \ell-k\}} \mathcal{J}_{(c_1-i, c_2-i, c_3-i)}.$$

Proof. As in the previous Lemma, we know that only triples $\mathbf{a} \in \mathbf{T}_{\mathbf{c}, \mathbf{c}}$ of the form $\mathbf{a} = (c_1 - i, c_2 - i, c_3 - i)$ with $i = \mathbf{a}(\mathbf{c}, \mathbf{c})$ can possibly contribute to $\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}})$. However, in this case our only restrictions are that $i \leq a_1 + c_2 - a_3 = k$ and $i \leq \min\{a_3 - a_1, a_3 - a_2\} = \ell - k$ so we need to include all such \mathbf{a} with $0 \leq i \leq \min\{k, \ell - k\}$. \square

Lemma 8.5. *Let $\mathbf{c} = \mathbf{d}$ and $i \leq \min\{k, \ell - k\}$. For $i \neq k$ we have*

$$\mathcal{J}_{(c_1-i, c_2-i, c_3-i)} = \begin{cases} 1 & \text{if } i = 0; \\ q - 2 & \text{if } i = 1; \\ (q - 1)^2 q^{i-2} & \text{if } i > 1 \end{cases}$$

whereas if $k \leq \ell - k$

$$\mathcal{J}_{(c_1-k, c_2-k, c_3-k)} = \begin{cases} q - 3 & \text{if } k = 1; \\ (q - 1)(q - 2)q^{k-2} & \text{if } k > 1. \end{cases}$$

Proof. Let $\mathbf{a} = (c_1 - i, c_2 - i, c_3 - i)$ and note that $a_1 + a_2 = a_3$ if and only if $i = k$.

First suppose that $i \neq k$ and so $a_1 + a_2 \neq a_3$. If $i = 0$ then $\mathbf{a}(\mathbf{c}, \mathbf{c}) = 0$ and $\mathbf{a} \notin \mathbf{T}_{\mathbf{c}_I, \mathbf{c}_J}$ for $I, J \subseteq S$ not both empty. This therefore implies that $\mathcal{J}_{\mathbf{a}} = |\mathbf{X}_{\mathbf{c}, \mathbf{c}}^{\mathbf{a}}| = 1$. For $i > 0$ we have $\mathbf{a}(\mathbf{c}, \mathbf{c}) = i$ and for $I, J \subseteq S$ not both empty $\mathbf{a} \in \mathbf{T}_{\mathbf{c}_I, \mathbf{c}_J}$ with $\mathbf{a}(\mathbf{c}_I, \mathbf{c}_J) = i - 1$. Consequently,

$$\mathcal{J}_{\mathbf{a}} = |\mathbf{X}_{\mathbf{c}, \mathbf{c}}^{\mathbf{a}}| + \sum_{\text{other } I, J} (-1)^{|I|+|J|} |\mathbf{X}_{\mathbf{c}_I, \mathbf{c}_J}^{\mathbf{a}}| = |\mathbf{X}_{\mathbf{c}, \mathbf{c}}^{\mathbf{a}}| - |\mathbf{X}_{\mathbf{c}_S, \mathbf{c}_S}^{\mathbf{a}}|.$$

When $i = 1$ we obtain $\mathcal{J}_{\mathbf{a}} = (q - 1) - 1 = q - 2$ whereas for $i > 1$ this gives $\mathcal{J}_{\mathbf{a}} = (q - 1)q^{i-1} - (q - 1)q^{i-2} = (q - 1)^2 q^{i-2}$.

Now suppose that $i = k$ and so $a_1 + a_2 = a_3$. Again we see that $\mathbf{a}(\mathbf{c}, \mathbf{c}) = i$ and $\mathbf{a} \in \mathbf{T}_{\mathbf{c}_I, \mathbf{c}_J}$ with $\mathbf{a}(\mathbf{c}_I, \mathbf{c}_J) = i - 1$ for $I, J \subseteq S$ not both empty. However, in this case $\mathbf{a}(\mathbf{c}_I, \mathbf{c}_J)' = i$ for $I \subseteq \{2\}$ and $J \subseteq \{1\}$ with $\mathbf{a}(\mathbf{c}_I, \mathbf{c}_J)' = i - 1$ otherwise. Thus

$$\begin{aligned} \mathcal{J}_{\mathbf{a}} &= |\mathbf{X}_{\mathbf{c}, \mathbf{c}}^{\mathbf{a}}| - |\mathbf{X}_{\mathbf{c}_{\{1\}}, \mathbf{c}}^{\mathbf{a}}| - |\mathbf{X}_{\mathbf{c}, \mathbf{c}_{\{2\}}}^{\mathbf{a}}| + |\mathbf{X}_{\mathbf{c}_{\{1\}}, \mathbf{c}_{\{2\}}}^{\mathbf{a}}| + \sum_{\text{other } I, J} (-1)^{|I|+|J|} |\mathbf{X}_{\mathbf{c}_I, \mathbf{c}_J}^{\mathbf{a}}| \\ &= |\mathbf{X}_{\mathbf{c}, \mathbf{c}}^{\mathbf{a}}| - |\mathbf{X}_{\mathbf{c}_{\{1\}}, \mathbf{c}_{\{2\}}}^{\mathbf{a}}|. \end{aligned}$$

When $k = 1$ this gives $\mathcal{J}_{\mathbf{a}} = (q - 1) - 2 = q - 3$ and when $k > 1$ we get $\mathcal{J}_{\mathbf{a}} = (q - 1)q^{k-1} - 2(q - 1)q^{k-2} = (q - 1)(q - 2)q^{k-1}$. \square

Lemma 8.6. *If $c_1 < d_1$ and $k = k' \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor$, then*

$$\mathcal{J}_{(c_1-k, d_2-k, c_3-k)} = \begin{cases} q-2 & \text{if } k=1; \\ (q-1)^2 q^{k-2} & \text{if } k>1. \end{cases}$$

Proof. Let $\mathbf{a} = (c_1 - k, d_2 - k, c_3 - k)$ and recall that $a_1 + a_2 \neq a_3$. Then $\mathbf{a}(\mathbf{c}, \mathbf{d}) = k$ and $\mathbf{a}(\mathbf{c}_I, \mathbf{d}_I) = k - 1$ for $I, J \subseteq S$ not both empty so the result follows by the argument for the first part of the previous Lemma. \square

Proof of Theorem 8.1. By Lemmas 8.4 and 8.5 we see that if $k = 1$ then

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) = \mathcal{J}_{(c_1, c_2, c_3)} + \mathcal{J}_{(c_1-1, c_2-1, c_3-1)} = 1 + (q-3) = q-2$$

and similarly when $1 < k \leq \ell - k$

$$\begin{aligned} \mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) &= \sum_{i=0}^k \mathcal{J}_{(c_1-i, c_2-i, c_3-i)} \\ &= 1 + (q-2) + (q-1)^2 q + \cdots + (q-1)^2 q^{k-3} + (q-1)(q-2)^{k-2} \\ &= (q-1)^2 q^{k-2}. \end{aligned}$$

However, if $\ell - k < k < \ell - 1$ then

$$\begin{aligned} \mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) &= \sum_{i=0}^{\ell-k} \mathcal{J}_{(c_1-i, c_2-i, c_3-i)} \\ &= 1 + (q-2) + (q-1)^2 q + \cdots + (q-1)^2 q^{\ell-k-3} + (q-1)^2 q^{\ell-k-2} \\ &= (q-1) q^{\ell-k-1}. \end{aligned}$$

and when $k = \ell - 1$, with $\ell > 2$,

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) = \mathcal{J}_{(c_1, c_2, c_3)} + \mathcal{J}_{(c_1-1, c_2-1, c_3-1)} = 1 + (q-2) = q-1.$$

Finally, if \mathbf{c} and \mathbf{d} have $c_3 = d_3$, $k = k'$ and $k \leq \lfloor \min\{\ell, \ell'\}/2 \rfloor$, then by the calculations above and Lemmas 8.4 and 8.6

$$\mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{d}}) = \mathcal{J}(V_{\mathbf{c}}, V_{\mathbf{c}}) = \mathcal{J}(V_{\mathbf{d}}, V_{\mathbf{d}}).$$

Hence $V_{\mathbf{c}}$ and $V_{\mathbf{d}}$ must be equivalent. \square

It should be noted that in proving the reducibility of $V_{\mathbf{c}}$, we have discovered a certain amount of information about its decomposition. Let $\mathbf{i} = (i, i, i)$ for $0 < i \leq \min\{k, \ell - k\}$ and consider the representation $V_{\mathbf{c}}^i = U_{\mathbf{c}}^i / \sum_{\mathbf{c} \prec \mathbf{d} \preceq \mathbf{c} - \mathbf{i}} U_{\mathbf{d}}^i$ where $U_{\mathbf{d}}^i = \text{Ind}_{C_{\mathbf{d}}}^{C_{\mathbf{c}-\mathbf{i}}} 1$. Clearly, $V_{\mathbf{c}} = \text{Ind}_{C_{\mathbf{c}-\mathbf{i}}}^K V_{\mathbf{c}}^i$ and we may use the results of Section 3 to show that

$$\mathcal{J}(V_{\mathbf{c}}^i, V_{\mathbf{c}}^i) = \sum_{I, J \subseteq S} (-1)^{|I|+|J|} |C_{\mathbf{c}_I} \setminus C_{\mathbf{c}-\mathbf{i}} / C_{\mathbf{c}_J}|.$$

However, the $(C_{\mathbf{c}_I}, C_{\mathbf{c}_J})$ -double coset representatives in $C_{\mathbf{c}-\mathbf{i}}$ are precisely the $t_{\mathbf{a}, x}$ in $R_{\mathbf{c}_I, \mathbf{c}_J}^1$ which have $\mathbf{a} \succeq \mathbf{c} - \mathbf{i}$. Thus

$$\mathcal{J}(V_{\mathbf{c}}^i, V_{\mathbf{c}}^i) = \sum_{\mathbf{a} \succeq \mathbf{c} - \mathbf{i}} \mathcal{J}_{\mathbf{a}},$$

where \mathcal{J}_a is as before, and Lemmas 8.4 and 8.5 immediately imply the following.

Proposition 8.7. *Let $0 < i \leq \min\{k, \ell - k\}$. For $i \neq k$ we have*

$$\mathcal{J}(V_{\mathbf{c}}^i, V_{\mathbf{c}}^i) = \begin{cases} (q-1) & \text{if } i = 1; \\ (q-1)q^{i-1} & \text{if } i > 1 \end{cases}$$

whereas if $k \leq \ell - k$

$$\mathcal{J}(V_{\mathbf{c}}^k, V_{\mathbf{c}}^k) = \begin{cases} (q-2) & \text{if } k = 1; \\ (q-1)^2 q^{k-2} & \text{if } k > 1. \end{cases}$$

In particular, this means that the irreducible constituents of $V_{\mathbf{c}}$ are induced from the irreducible constituents of $V_{\mathbf{c}}^{\min\{k, \ell-k\}}$.

9. APPLICATION TO STEINBERG REPRESENTATIONS

In [5], Lees defined a virtual representation S_r of $\mathrm{GL}(n, \mathcal{R}/\mathcal{P}^r)$ which possessed properties that were similar to those of the Steinberg representation of $\mathrm{GL}(n, \mathbf{f})$. Further, he stated without proof that S_r was in fact a subrepresentation of the permutation representation over the subgroup of upper triangular matrices. Although this is the case for $n = 2$ and for $n = 3$ with $r \leq 2$, we will show that S_r is not a true representation for $r > 2$.

When $n = 3$, and pulling back to K , the expression of S_r as an alternating sum of permutation representations reduces to

$$(20) \quad [S_r] = \sum_{c_1, c_2=0}^r (-1)^{c_1+c_2} [U_{(c_1, c_2, \max\{c_1, c_2\})}].$$

If $r = 0$ then we obtain the trivial representation $S_0 = V_{(0,0,0)}$ and $r = 1$ produces $S_1 = V_{(1,1,1)}$, the Steinberg representation of $\mathrm{GL}(3, \mathbf{f})$ pulled back to K . More generally, S_r can be constructed inductively in the following way.

Lemma 9.1. *Let $r \geq 2$, then*

$$(21) \quad [S_r] = [S_{r-2}] + \sum_{\mathbf{c}} (-1)^{r-c_1} [V_{\mathbf{c}}]$$

where the sum runs over all triples $\mathbf{c} = (c_1, c_2, r) \in \mathbf{T}$ with $c_1 \equiv c_2 \pmod{2}$.

Proof. This can easily be seen by comparing the coefficients of $U_{\mathbf{c}}$ in (20) and (21) for each $\mathbf{c} \preceq (r, r, r)$. \square

In particular, Lemma 9.1 implies that

$$[S_2] = [V_{(2,2,2)}] - [V_{(1,1,2)}] + [V_{(0,2,2)}] + [V_{(2,0,2)}] + [V_{(0,0,0)}].$$

Further, $V_{(1,1,2)}$, $V_{(0,2,2)}$ and $V_{(2,0,2)}$ are all equivalent so this is actually the sum of three irreducible representations $S_2 \simeq V_{(2,2,2)} + V_{(1,1,2)} + V_{(0,0,0)}$. However, when $r > 2$ we see that $V_{(r-1, r-1, r)}$ still appears with coefficient -1 in (21) and in this case $V_{(r-1, r-1, r)}$ has no constituents in common with any other component $V_{\mathbf{c}}$. Hence it cannot cancel with any other term in (21) and we have shown the following.

Proposition 9.2. *S_r is a true representation if and only if $r \leq 2$.*

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